

**POLYNOMIAL HULLS AND H^∞ CONTROL
FOR A HYPOCONVEX CONSTRAINT**

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ABSTRACT. We say that a subset of \mathbf{C}^n is hypoconvex if its complement is the union of complex hyperplanes. Let Δ be the closed unit disk in \mathbf{C} , $\Gamma = \partial\Delta$. We prove two conjectures of Helton and Marshall. Let ρ be a smooth function on $\Gamma \times \mathbf{C}^n$ whose sublevel sets have compact hypoconvex fibers over Γ . Then, with some restrictions on ρ , if Y is the set where ρ is less than or equal to 1, the polynomial convex hull of Y is the union of graphs of analytic vector valued functions with boundary in Y . Furthermore, we show that the infimum $\inf_{f \in H^\infty(\Delta)^n} \|\rho(z, f(z))\|_\infty$ is attained by a unique bounded analytic f which in fact is also smooth on Γ . We also prove that if ρ varies smoothly with respect to a parameter, so does the unique f just found.

We address two conjectures of Helton and Marshall from [HMa, p. 183] which generalize previous theorems regarding an H^∞ control problem over the disk and polynomial hulls of compact sets in \mathbf{C}^{n+1} fibered over the circle in \mathbf{C} .

If Y is a compact set in \mathbf{C}^n , then the *polynomial (convex) hull* \widehat{Y} of Y is given by

$$\widehat{Y} = \{z \in \mathbf{C}^n \mid |P(z)| \leq \sup_{w \in Y} |P(w)| \text{ for all polynomials } P \text{ on } \mathbf{C}^n\}.$$

Let Δ be the closed unit disk in \mathbf{C} , let Γ be the unit circle, let $\Pi : \Delta \times \mathbf{C} \rightarrow \Delta$ be projection, and let $\rho : \Gamma \times \mathbf{C}^n \rightarrow \mathbf{R}$ be C^2 . Let $D^i\rho =$ the i^{th} derivative of ρ , (thought of as an i -linear mapping, as in [L].) Then $D\rho(z, w)$ and $D^2\rho(z, w)$ are the gradient and real Hessian at (z, w) , respectively, for $(z, w) \in \Gamma \times \mathbf{C}^n$. Let $D_w\rho(z, w)$ be the vector $(\frac{\partial\rho}{\partial w_1}(z, w), \frac{\partial\rho}{\partial w_2}(z, w), \dots, \frac{\partial\rho}{\partial w_n}(z, w))$ and let $D_{\overline{w}}\rho(z, w)$ denote its conjugate. Note that $D_{\overline{w}}\rho$ is the complex form for the gradient of ρ in w . In this work, if L, B are linear and bilinear maps, respectively, we shall write $L[u]$ and $B[u][v]$ to denote the values of L and B . Where we encounter functions which are not linear, we shall use parentheses instead of brackets to denote values.

Following [HMa], we define ρ to be *hypoconvex* if there exists a $\mathcal{K} \geq 0$ such that for every point $(z, w_1, w_2, \dots, w_n) \in \Gamma \times \mathbf{C}$ such that if

$$\sum_{j=1}^n u_j \frac{\partial \rho}{\partial w_j}(z, w_1, w_2, \dots, w_n) = 0$$

we have

$$(1) \quad D^2 \rho(z, w_1, w_2, \dots, w_n)[0, u_1, u_2, \dots, u_n][0, u_1, u_2, \dots, u_n] \geq \mathcal{K} \|u\|^2,$$

where $\|\cdot\|$ is the standard Euclidean norm. Let $K^t = \{(z, w_1, w_2, \dots, w_n) \in \Gamma \times \mathbf{C}^n \mid \rho(z, w_1, w_2, \dots, w_n) = t\}$ and let $K_z^t = \{(w_1, w_2, \dots, w_n) \mid (z, w_1, w_2, \dots, w_n) \in K^t\}$ be the fiber of K^t over z . If $\mathcal{K} > 0$ then (1) says that on the complex tangent space of the fibers, $D^2 \rho$ is positive definite, so the complex hyperplane in \mathbf{C}^n tangent to the set K_z^t is locally external to the set where ρ is less than or equal to t .

In [Hö], a set in \mathbf{C}^n is defined to be *linearly convex* if the complement of the set is the union of complex $(n-1)$ -dimensional affine planes. The notion of such sets appeared in 1935 under the name “planarkonvex” in work of Behnke and Peschl [BP] (although the notion is slightly weaker), again in 1940 in [BS], later discussed in the 1952 dissertation [St] of Strehlke, and then reappeared in the 1960’s in work of Martineau, who used the term “linéellement convexe” (see [M]). Kiselman (see [Ki]) uses the term “lineal convexity” and Vityaev (see [V]), “complex geometric convexity.” In order to avoid confusion with ordinary convexity, we shall use the word *hypoconvex* instead of *linearly convex*. We also feel the word *hypoconvex* is suggestive of the geometry of the situation, since this notion is somewhat weaker than ordinary convexity. However, the reader should be aware of the fact that we are not following the majority of the literature in using this terminology.

In summary, we shall call a set in \mathbf{C}^n *hypoconvex* if its complement is the union of complex $(n-1)$ -dimensional affine planes and we shall call ρ *hypoconvex* on an open subset U of $\Gamma \times \mathbf{C}$ if it satisfies (1) in U . If for every compact subset of U there exists a $\mathcal{K} > 0$ such that (1) holds, we shall say that ρ is *strictly hypoconvex* on U .

A C^2 -bounded hypoconvex set whose defining function has real Hessian which is positive definite on the complex tangent spaces at every boundary point of the set shall be called *strictly hypoconvex*. Thus K_z^t bounds a strictly hypoconvex compact set for all $z \in \Gamma$.

We shall work under the assumption that $n \geq 2$ but our arguments work for $n = 1$ with minor adjustments.

Our first plan is to prove (in Theorem 2) that a compact set K in $\Gamma \times \mathbf{C}^n$ with smoothly bounded connected strictly hypoconvex fibers containing the origin has a polynomial hull which is the union of the graphs of analytic vector-valued functions over the closed unit disk, provided that the set can be deformed in a reasonable manner to a compact set whose fibers are balls. This has been proven provided K has convex fibers in [AW] and [S3], and if $n = 1$ for compact K with connected and simply connected fibers in [F1] and [S2], and also in [HMa]. Our result then generalizes both, since convex sets in \mathbf{C}^n are hypoconvex, and any subset of \mathbf{C} is hypoconvex. There exist examples of compact sets in \mathbf{C}^n fibered over the circle with contractible fibers whose polynomial hulls are not the union of graphs over the disk; see [HMe1] and [Če]. We say that the function f defined on Γ is a *selector* for the set $K \subset \Gamma \times \mathbf{C}^n$ if $f(z) \in K_z$ for all $z \in \Gamma$.

A problem of H^∞ control is to compute

$$(2) \quad \gamma_\rho \equiv \inf_{f \in H^\infty(\Delta)^n} \operatorname{ess\,sup}_{z \in \Gamma} \rho(z, f(z)),$$

using notation from [HMa]. We shall call γ_ρ the *optimal control* for ρ . It is also of interest to determine various facts about an f which attains this minimax: whether it exists, is unique, is smooth, and whether it possesses other properties to be mentioned later. Such an f we call a “solution” to the H^∞ control problem (2). We shall prove (in Theorem 3) that for strictly hypoconvex ρ the H^∞ control problem is uniquely solvable with a vector valued function which is smooth, again subject to restrictions to be described. This has been done in some important cases in [HMa],[Hu] and [HV]. [HV] uses methods similar to the ones we employ and

proves related results. In [V], Vityaev shows that if ρ is strictly hypoconvex where $\rho = \gamma_\rho$, then if a solution to (2) is smooth on Γ , it is the only smooth solution.

We also show in Theorem 4 that if ρ changes smoothly with respect to a parameter, then so does γ_ρ and the solution to (2).

We shall make the following assumptions on ρ throughout our work:

- (3) $\left[\begin{array}{l} \text{(a) } \rho : \Gamma \times \mathbf{C}^n \rightarrow [0, \infty) \text{ is continuous, and } C^6\text{-smooth where } \rho \neq 0; \\ \text{(b) there exists an } R > 1 \text{ such that if } 0 < \rho(z, w) \leq R \text{ then } D_w \rho(z, w) \neq \\ 0 \text{ and } \rho \text{ is strictly hypoconvex as in (1) where } \rho \text{ is smooth;} \\ \text{(c) for every } t, 0 < t \leq R, \text{ the set } K^t \text{ where } \rho = t \text{ is compact, with fibers} \\ K_z^t \text{ diffeomorphic to } \partial B_n, \text{ where } B_n \text{ is the open unit ball in } \mathbf{C}^n; \\ \text{(d) } K = K^1; \\ \text{(e) } \{(z, w) \mid \rho(z, w) = R\} = \{(z, w) \mid |w| = R\} \text{ and } \rho(z, w) > R \text{ if } |w| > R; \\ \text{(f) There exists a continuous function } S(z), |S(z)| < R, \text{ such that} \\ \{(z, w) \in \Gamma \times \mathbf{C}^n \mid w = S(z)\} = \{(z, w) \in \Gamma \times \mathbf{C}^n \mid \rho(z, w) = 0\}. \end{array} \right.$

From Theorem 1 of [YK], we may conclude that if $0 < t < R$, and D_z^t is the closed domain enclosed by the level set $\{w \mid \rho(z, w) = t\}$, then the complex tangent spaces to boundary points of D_z^t do not meet $\text{int } D_z^t$. See also [Ki]. By Corollary 4.6.9 of [Hö] applied to $\text{int } D_z^t$, we may conclude that $\text{int } D_z^t$ is hypoconvex, so D_z^t is as well, as the intersection of D_z^s for $s > t$. D_z^t is also clearly strictly hypoconvex. From the same corollary, we obtain that $\text{int } D_z^t$ is “ \mathbf{C} -convex”, i.e., if P is a 1-dimensional complex affine subspace of \mathbf{C}^n which meets $\text{int } D_z^t$, then the intersection is a connected and simply connected subset of P . If $P \cap \text{int } D_z^t \neq \emptyset$, then the intersection is a smoothly bounded subset of P , because P cannot be a tangent to D_z^t ; hence the derivative of ρ restricted to P where $\rho = t$ is not identically zero and $P \cap D_z^t$ is the closure of $P \cap \text{int } D_z^t$. If $P \cap \text{int } D_z^t = \emptyset$ but P does meet D_z^t then we claim it only meets D_z^t in one point. Such a P must be tangent to the boundary of D_z^t at any intersection point. As observed earlier, the tangent space is locally disjoint from D_z^t near such a point. If there are two such points, then

we may perturb P slightly and obtain a P' whose intersection with $\text{int } D_z^t$ is not connected. Thus the intersection of any complex hyperplane with D_z^t is either a point or a Jordan domain with boundary as smooth as ρ . We note that $D_z^t \subset \widehat{K}_z^t$. The reverse inclusion also holds: Proposition 1 of [Z] shows that the interior of D_z^s is polynomially convex for all s , so D_z^t is polynomially convex as the intersection of the polynomially convex open sets D_z^s , $s > t$. Thus $D_z^t = \widehat{K}_z^t$.

§1 A perturbation theorem.

Let $\langle \cdot, \cdot \rangle$ be the complex inner product on \mathbf{C}^n , $\langle a, b \rangle = \sum_{j=1}^n a_j \overline{b_j}$. We shall allow the arguments a and b to be functions whose values are in \mathbf{C}^n ; then the operation $\langle \cdot, \cdot \rangle$ is pointwise inner product of the two functions. Let $A(\Delta) = \{f : \Delta \rightarrow \mathbf{C} \mid f \text{ is continuous on } \Delta \text{ and analytic in } \text{int } \Delta\}$, $L_C^2(\Gamma) =$ the set of square integrable complex valued functions on Γ under the ordinary inner product $\langle f, g \rangle_2 = \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi}$, $W^{1,2}(\Gamma) =$ the Sobolev space of complex functions on Γ with first derivatives in θ in $L_C^2(\Gamma)$, under the real inner product $\langle f, g \rangle_{1,2} = C \text{Re} \langle f, g \rangle_2 + \text{Re} \langle \frac{\partial f}{\partial \theta}, \frac{\partial g}{\partial \theta} \rangle_2$, where C is a large positive constant to be chosen later. Let $W_R^{1,2}(\Gamma) =$ the real valued elements in $W^{1,2}(\Gamma)$, $H^2(\Delta) =$ the elements of $L_C^2(\Gamma)$ whose negative Fourier coefficients vanish and $H^{1,2}(\Delta) = H^2(\Delta) \cap W^{1,2}(\Gamma)$. Then $W^{1,2}(\Gamma) \subset C(\Gamma)$ is a continuous inclusion by the Sobolev embedding theorem; but we can show quickly why this holds. If $f(z) = \sum_{j=-\infty}^{\infty} a_j z^j$ is in $W^{1,2}(\Delta)$ then $\|f_\theta\|_2^2 = \sum_{j=-\infty}^{\infty} j^2 |a_j|^2 < \infty$ so $\sum_{j=-\infty}^{\infty} |a_j| \leq |a_0| + (\sum_{j=-\infty}^{\infty} j^2 |a_j|^2)^{\frac{1}{2}} (2 \sum_{j=1}^{\infty} \frac{1}{j^2})^{\frac{1}{2}} = |a_0| + \frac{\pi}{\sqrt{3}} \|f_\theta\|_2$. Thus the Fourier series of f converges absolutely and uniformly on Γ and its supremum norm is bounded by $\sum_{j=-\infty}^{\infty} |a_j| \leq |a_0| + \frac{\pi}{\sqrt{3}} \|f_\theta\|_2$, which is an equivalent norm for $W^{1,2}$. Then $f \in A(\Delta)$ and has small supremum norm if $\|f\|_{1,2}$ is small. We let $H_0^{1,2}(\Delta) =$ the set of elements of $H^{1,2}(\Delta)$ which have value zero when $z = 0$.

In the following we use techniques similar to those of Forstnerič [F1]. We begin with the graph of an analytic vector valued function f which is extremal in a sense that will be clear later (it will be in the boundary of a particular polynomial hull)

and find graphs close by which are similarly extremal. We spell out conditions we shall require of such graphs and use the implicit function theorem on Banach spaces to establish their existence and uniqueness.

Theorem 1. *Let ρ satisfy (3) with S identically zero. Suppose that there exist $f, g \in H^{1,2}(\Delta)^n$ such that $f, g \in C^4(\Gamma)$, $\rho(z, f(z))$ is constant in z , say $= 1$, $\sum_{j=1}^n f_j(z)g_j(z) = 1$ and the affine complex tangent plane to K_z^1 at $(z, f(z))$ is $\{(w_1, w_2, \dots, w_n) \in \mathbf{C}^n \mid \sum_{j=1}^n g_j(z)w_j = 1\}$. Then for some neighborhood $N(f(0))$ in \mathbf{C}^n , there exist C^1 maps $F, G : N(f(0)) \rightarrow H^{1,2}(\Delta)^n$, $F = (F_1, F_2, \dots, F_n)$, $G = (G_1, G_2, \dots, G_n)$, such that $\rho(z, F(w)(z))$ is constant in $z \in \Gamma$ for fixed $w \in N(f(0))$, $F(w)(0) = w$, $\sum_{j=1}^n F_j(w)(z)G_j(w)(z) = 1$, and the complex tangent plane to $K_z^{\rho(z, F(w)(z))}$ at $F(w)(z)$ is $\{(w_1, w_2, \dots, w_n) \in \mathbf{C}^n \mid \sum_{j=1}^n G_j(w)(z)w_j = 1\}$.*

Remark. For Theorem 1, it suffices to assume that ρ is only C^4 and that the set where ρ equals R bounds a set which is merely strictly convex instead of being a ball.

Proof. Condition (3)(f) with $S = 0$ guarantees that K_z^t separates the origin in \mathbf{C}^n from the point at infinity for $t > 0$.

We first reduce to the case where $g(z) = (1, 0, 0, \dots, 0)$ and f_1 , the first coordinate of f , is identically 1. To do this, we construct an $n \times n$ matrix $M(z)$ of analytic functions such that the first column is given by $g(z)$. By [SW], Theorem 2.1, since $g(z)$ is never zero on Δ , there exist analytic \mathbf{C}^n -valued functions h_1, h_2, \dots, h_{n-1} in $A(\Delta)^n$ such that $g(z)$ along with the h_i generate $A(\Delta)^n$ as a module over $A(\Delta)$. In particular, if the last $n - 1$ columns of M are h_1, h_2, \dots, h_{n-1} , then for all $z \in \Delta$, the determinant of $M(z)$ is nonzero. We may mollify the h_i so slightly they are also in $C^4(\Gamma)$ but the determinant of $M(z)$ is still nonzero. Then consider the C^4 function given by $\tilde{\rho}(z, w) = \rho(z, w \cdot M(z)^{-1})$, regarding w as a row vector. Then we get corresponding sets $\tilde{K}^t \equiv \{(z, w) \in \Gamma \times \mathbf{C}^n \mid \tilde{\rho}(z, w) = t\}$ such that under $w \mapsto w \cdot M(z)$, the complex tangent space to K_z at $f(z)$ is mapped to the complex tangent space to \tilde{K}_z at $(1, 0, 0, \dots, 0)$, which is $\{(w_1, w_2, \dots, w_n) \mid w_1 = 1\}$. Let $\tilde{g}(z) =$

$(1, 0, 0, \dots, 0)$. We also note that we get an $\tilde{f}(z) \equiv f(z) \cdot M(z)$ corresponding to $f(z)$ such that $\tilde{f}_1(z) = 1$. Then the conditions for the theorem are satisfied with $\tilde{\rho}$, \tilde{f} and \tilde{g} , except that the conditions of the Remark above hold. The linearity and invertibility of $w \mapsto w \cdot M(z)$ in w guarantees that $\tilde{\rho}$ is strictly hypoconvex. Once we have Theorem 1 proven for $\tilde{\rho}$, obtaining $\tilde{F}(w), \tilde{G}(w), \tilde{N}(\tilde{f}(0))$ with the desired properties, we define $F(w)(z) = \tilde{F}(w \cdot M(0))(z) \cdot M(z)^{-1}$. Then F is defined in a neighborhood $N(f(0))$ of $f(0)$ since $\tilde{f}(0) = f(0) \cdot M(0)$. We have $\rho(z, F(w)(z)) = \rho(z, \tilde{F}(w \cdot M(0))(z) \cdot M(z)^{-1}) = \tilde{\rho}(z, \tilde{F}(w \cdot M(0))(z))$ which is constant in z for fixed w in some neighborhood of $f(0)$. If we write $v = w \cdot M(z)$, then $\frac{\partial \rho}{\partial w_j} = \sum_{i=1}^n (\frac{\partial \tilde{\rho}}{\partial v_i} \frac{\partial v_i}{\partial w_j} + \frac{\partial \tilde{\rho}}{\partial \bar{v}_i} \frac{\partial \bar{v}_i}{\partial w_j}) = \sum_{i=1}^n \frac{\partial \tilde{\rho}}{\partial v_i} \frac{\partial v_i}{\partial w_j}$. Let x^T denote the transpose of x . Now $\frac{\partial v_i}{\partial w_j} = M_{ji}$, so $D_w \rho(z, F(w)(z))^T = M(z) (D_w \tilde{\rho}(z, \tilde{F}(w \cdot M(0))(z)))^T = a(w)(z) M(z) \cdot (\tilde{G}(w \cdot M(0))(z))^T$, where $a(w)$ is C^1 in w and we claim the winding number of $a(w)$ is zero in z . If $w = f(0)$ then this is merely the statement that $\sum_{j=1}^n \tilde{f}_j(z) \frac{\partial \tilde{\rho}}{\partial w_j}(z, \tilde{f}(z))$ has winding number zero. This function is not zero for $z \in \Gamma$ because the complex tangent planes to the fibers of \tilde{K}^t never pass through the origin. Now we may deform \tilde{f} through a homotopy $\{\tilde{f}^t\}$, $1 \leq t \leq R$ so that \tilde{f}^t is continuous, $\tilde{f}^1 = \tilde{f}$, $\rho(z, \tilde{f}^t(z)) = t$ and $\sum_{j=1}^n \tilde{f}_j^t(z) \frac{\partial \tilde{\rho}}{\partial w_j}(z, \tilde{f}^t(z))$ has the same winding number as $\sum_{j=1}^n \tilde{f}_j(z) \frac{\partial \tilde{\rho}}{\partial w_j}(z, \tilde{f}(z))$; the former is never zero for $z \in \Gamma$ for the same reason as stated above for $\sum_{j=1}^n \tilde{f}_j(z) \frac{\partial \tilde{\rho}}{\partial w_j}(z, \tilde{f}(z))$. Then \tilde{K}^R can be deformed smoothly out to some $\tilde{K}^{R'}$ which has spherical fibers of radius \tilde{R}' such that \tilde{K}^t has convex fibers for $R \leq t \leq R'$. Deforming \tilde{f}^R similarly out to $\tilde{f}^{R'}$ such that $\rho(z, \tilde{f}^t(z)) = t$ for $R \leq t \leq R'$, we find that $\sum_{j=1}^n \tilde{f}_j^{R'}(z) \frac{\partial \tilde{\rho}}{\partial w_j}(z, \tilde{f}^{R'}(z)) = |\tilde{f}^{R'}(z)|^2 = R'^2$, a constant function (with winding number 0). By the homotopy to $\sum_{j=1}^n \tilde{f}_j(z) \frac{\partial \rho}{\partial w_j}(z, \tilde{f}(z))$, $a(v)(z)$ has winding number 0 for $v = f(0)$ and for all v in $N(f(0))$. Then $D_w \rho(z, F(v)(z))$ is a scalar function $a(v)(z)$ times an analytic vector function $b(v)(z)$ which is C^1 in v , so $\sum_{j=1}^n F_j(v)(z) b_j(v)(z)$ is analytic and equals $\frac{1}{a(v)(z)}$ times $\sum_{j=1}^n F_j(v)(z) \frac{\partial \rho}{\partial w_j}(z, F(v)(z))$, both of which have winding number zero. (The latter function has winding number zero by an argument in K^t similar

to the one just given in \tilde{K}^t .) We can then define

$$G(w)(z) \equiv \frac{b(w)(z)}{\sum_{j=1}^n F_j(w)(z)b_j(w)(z)}$$

for w in $N(f(0))$. Then $F(w)$ and $G(w)$ satisfy the properties required. This proves Theorem 1 provided we prove it in the case where $g(z) = (1, 0, 0, \dots, 0)$ and $f_1 = 1$.

We note that there exists a function $h \in H^{1,2}(\Delta)$ such that h is nonzero for $z \in \Delta$ and $\sum_{j=0}^n f(z) \frac{\partial \rho}{\partial w_n}(z, f(z)) = \frac{\partial \rho}{\partial w_1}(z, f(z))$ has the same argument as $h(z)$ on Γ . To obtain h , we need the fact that this function is nonzero on Γ , with winding number 0, using an argument from the previous paragraph. We also need that the harmonic conjugation operator is continuous on $W^{1,2}$, that if $f \in W^{1,2}$ then so is e^f , and that $\log(f)$ is as well if f is nonzero and has winding number 0.

Suppose that $u, v \in W_R^{1,2}(\Gamma)$. Let \tilde{u} denote the harmonic conjugate of u whose value at 0 is 0. Let $H^{1,2}(\Delta)^{n-1}$ be the subspace of $H^{1,2}(\Delta)^n$ of n -tuples $k = (k_1, k_2, \dots, k_n)$ such that $k_1 = 0$. By $W_R^{1,2}(\Gamma)/\mathbf{R}$ we mean the quotient of $W_R^{1,2}$ by the real constant functions.

As in [F1], let $X(z)$ denote an element of $H^{1,2}(\Delta)$ which points in the same real direction as the outward normal to K_z^1 at 1, which is $\frac{\partial \rho}{\partial w_1}(z, f(z))$. (In fact, we can take $X = 1/h$.) Then $\text{wind } X(z) = 0$, so X has no zeroes on the closed disk. Let

$$F(u, v, k) = f + (u + \tilde{u}i)fX + v(0)ifX + k$$

$$G(l) = g + l$$

and $F = (F_1, F_2, \dots, F_n)$, $G = (G_1, G_2, \dots, G_n)$. Also let $w_0 = f(0)$.

Consider the mapping $\Phi : W_R^{1,2}(\Gamma) \times W_R^{1,2}(\Gamma) \times H^{1,2}(\Delta)^{n-1} \times H^{1,2}(\Delta)^n \times \mathbf{C}^n \rightarrow W_R^{1,2}(\Gamma)/\mathbf{R} \times W^{1,2}(\Gamma)^n \times \mathbf{C}^n$, where

$$\Phi(u, v, k, l, w) = (\Phi_1(u, v, k, l, w), \Phi_2(u, v, k, l, w) - \Phi_3(u, v, k, l, w), \Phi_4(u, v, k, l, w))$$

and

$$\Phi_1(u, v, k, l, w)(\cdot) = \rho(\cdot, F(u, v, k)(\cdot)) + \mathbf{R},$$

$$\Phi_2(u, v, k, l, w)(z) = \frac{1 + v(z) + i\tilde{v}(z)}{\sum_{j=1}^n \overline{(F_j(u, v, k)(z))} \left(\frac{\partial \rho}{\partial w_j}(z, F(u, v, k)(z)) \right)} \bullet$$

$$\left(\frac{\partial \rho}{\partial w_1}(z, F(u, v, k)(z)), \frac{\partial \rho}{\partial w_2}(z, F(u, v, k)(z)), \dots, \frac{\partial \rho}{\partial w_n}(z, F(u, v, k)(z)) \right),$$

$$\Phi_3(u, v, k, l, w)(z) = (\overline{G_1(l)}(z), \overline{G_2(l)}(z), \dots, \overline{G_n(l)}(z)), \text{ and}$$

$$\Phi_4(u, v, k, l, w) = \int_0^{2\pi} F(u, v, k)(e^{i\theta}) \frac{d\theta}{2\pi} - w.$$

Note that $\Phi(0, 0, 0, 0, w_0) = (0, 0, 0)$. In §5, we show that Φ is a C^1 map near $(0, 0, 0, 0, w_0)$. We claim that the partial derivative of Φ in (u, v, k, l) when $(u, v, k, l, w) = (0, 0, 0, 0, w_0)$ is an invertible mapping. We denote this partial derivative by $D_{(u,v,k,l)}$. Using the implicit function theorem, we will be able to make the conclusions of the theorem. The reader is invited to look at the end of the proof to see how this happens. We also note that our technique resembles that used in [L2] and [BL].

Following Forstnerič [F1], we compute $D\Phi_1(0, 0, 0, 0, w_0)[u, 0, 0, 0, 0] = 2\text{Re}(\sum_{j=1}^n \frac{\partial \rho}{\partial w_j}(\cdot, f(\cdot))(u + \tilde{u}i)f_j X) + \mathbf{R} = 2\text{Re}(\frac{\partial \rho}{\partial w_1}(\cdot, f(\cdot))(u + \tilde{u}i)X) + \mathbf{R}$. Since $X(z)$ has the same argument as $\frac{\partial \rho}{\partial w_1}(z, f(z))$, $2\text{Re}(\frac{\partial \rho}{\partial w_1}(\cdot, f(\cdot))(u + \tilde{u}i)(X) = 2\frac{\partial \rho}{\partial w_1}(\cdot, f(\cdot))X \text{Re}(u + \tilde{u}i) = 2\frac{\partial \rho}{\partial w_1}(\cdot, f(\cdot))Xu$. Then $I \equiv 2\frac{\partial \rho}{\partial w_1}(\cdot, f(\cdot))X$ is a positive $W_R^{1,2}$ function.

We proceed with several steps, initially showing that $D_{(u,v,k,l)}\Phi(0, 0, 0, 0, w_0)$ is injective in (u, v, k, l) . First we note that $D\Phi_1(0, 0, 0, 0, w_0)[0, v, k, l, 0] = 0$ for all $v \in W_R^{1,2}(\Gamma)$, $k \in (H^{1,2}(\Delta))^{n-1}$ and $l \in (H^{1,2}(\Delta))^n$. This is obvious with l . For k and v we see that $D\Phi_1(0, 0, 0, 0, w_0)[0, v, k, 0, 0](z) = D\rho(z, f(z))[v(0)if(z)X(z) + k(z)] = 0$, since $v(0)if(z)X(z) + k(z)$ is a tangent to K_z at $(z, f(z))$.

We shall have need for the fact that $\langle D\Phi_2(0, 0, 0, 0, w_0)[0, 0, k, 0, 0], f \rangle = 0$. To see this, we first note that $\langle \Phi_2(u, v, k, l, w_0), F(u, v, k) \rangle = 1 + v + \tilde{v}i$. Differentiating both sides with respect to k at $(u, v, k, l, w) = (0, 0, 0, 0, w_0)$, we find that

$$\langle D\Phi_2(0, 0, 0, 0, w_0)[0, 0, k, 0, 0], F(0, 0, 0) \rangle + \langle \Phi_2(0, 0, 0, 0, w_0), DF(0, 0, 0)[0, 0, k] \rangle = 0.$$

Now $DF(0, 0, 0)[0, 0, k] = k$, $F(0, 0, 0) = f$, $\Phi_2(0, 0, 0, 0, w_0) = (1 + v + \tilde{v}i)\bar{g}$ and $\langle \bar{g}, k \rangle = 0$ so the above simplifies to

$$(4) \quad \langle D\Phi_2(0, 0, 0, 0, w_0)[0, 0, k, 0, 0], f \rangle = 0,$$

as desired.

Clearly $D\Phi_2(0, 0, 0, 0, w_0)[0, 0, 0, l, 0] = 0$ and $D\Phi_3(0, 0, 0, 0, w_0)[0, 0, k, 0, 0] = 0$ for all $k \in H^{1,2}(\Delta)^{n-1}$ and $l \in H^{1,2}(\Delta)^n$, as Φ_2 is constant in l and Φ_3 is constant in k .

Suppose that $D\Phi(0, 0, 0, 0, w_0)[u, v, k, l, 0] = 0$. Then uI is a constant c , where I is a nonzero real $W^{1,2}$ function. In order for $D\Phi_4(0, 0, 0, 0, w_0)[u, v, k, l, 0]$ to equal zero, we must have $0 = \operatorname{Re}\langle D\Phi_4(0, 0, 0, 0, w_0)[u, v, k, l, 0], X(0)\overline{g(0)} \rangle = u(0)|X(0)|^2$. Since X has winding number zero, we have $u(0) = 0$. Hence $c((\frac{1}{I})(0)) = 0$, where $(\frac{1}{I})(0)$ denotes the value at 0 of the harmonic extension of $1/I$ to the closed disk. Now $(\frac{1}{I})(0) \neq 0$ since I is positive, so we must have $c = 0$. Thus $u = 0$. From the fact that $D\Phi_4(0, 0, 0, 0, w_0)[0, v, k, l, 0] = 0$, we also find that $v(0) = 0$ ($0 = \langle D\Phi_4(0, 0, 0, 0, w_0)[0, v, k, l, 0], X(0)\overline{g(0)} \rangle = v(0)|X(0)|^2i$) and $k(0) = \int_0^{2\pi} k(e^{i\theta}) d\theta = \vec{0}$, where $\vec{0}$ = the origin in \mathbf{C}^n .

Then the derivative of the middle coordinate of Φ must be 0 in the direction of $[0, v, k, l, 0]$, so

$$\begin{aligned} 0 &= D\Phi_2(0, 0, 0, 0, w_0)[0, v, k, l, 0] - D\Phi_3(0, 0, 0, 0, w_0)[0, v, k, l, 0] \\ &= D\Phi_2(0, 0, 0, 0, w_0)[0, v, k, 0, 0] - D\Phi_3(0, 0, 0, 0, w_0)[0, 0, 0, l, 0], \end{aligned}$$

making use of the previously observed facts that $D\Phi_2(0, 0, 0, 0, w_0)$ does not depend on l and $D\Phi_3(0, 0, 0, 0, w_0)$ does not depend on v, k . Thus

$$\langle D\Phi_2(0, 0, 0, 0, w_0)[0, v, k, 0, 0], f \rangle = \langle D\Phi_3(0, 0, 0, 0, w_0)[0, 0, 0, l, 0], f \rangle.$$

From (4), we get

$$\langle D\Phi_2(0, 0, 0, 0, w_0)[0, v, 0, 0, 0], f \rangle = \langle D\Phi_3(0, 0, 0, 0, w_0)[0, 0, 0, l, 0], f \rangle,$$

so then $1 + v + \bar{v}i = (1 + v + \bar{v}i)\langle \bar{g}, f \rangle = \langle D\Phi_2(0, 0, 0, 0, w_0)[0, v, 0, 0, 0], f \rangle = \langle D\Phi_3(0, 0, 0, 0, w_0)[0, 0, 0, l, 0], f \rangle = \sum_{j=1}^n \bar{l}_i f_i$. (Note that we need the fact that $v(0) = 0$ implies $DF(0, 0, 0)[0, v, 0] = 0$.) Thus $v + \bar{v}i = 0$ (as an analytic and conjugate analytic function whose value at 0 is 0.)

So $\langle D\Phi_2(0, 0, 0, 0, w_0)[0, 0, k, 0, 0], k \rangle = \langle D\Phi_3(0, 0, 0, 0, w_0)[0, 0, 0, l, 0], k \rangle$. Now suppose k is not identically 0; then

$$\begin{aligned} & \langle D\Phi_2(0, 0, 0, 0, w_0)[0, 0, k, 0, 0], k \rangle(z) \\ &= \left\langle \frac{D^2\rho(z, f(z))[k(z)]}{\sum_{j=1}^n (\overline{f_j(z)}) \left(\frac{\partial \rho}{\partial \overline{w_j}}(z, f(z))\right)}, k(z) \right\rangle \\ & \quad + C(z) \sum_{j=1}^n \frac{\partial \rho}{\partial \overline{w_1}}(z, f(z)) \overline{k_j}(z) \\ & \quad (\text{using the fact that } v = 0) \\ &= \left\langle \frac{D^2\rho(z, f(z))[k(z)]}{\frac{\partial \rho}{\partial \overline{w_1}}(z, f(z))}, k(z) \right\rangle \end{aligned}$$

since $k(z)$ is a complex tangent to K_z at $f(z)$. However we also have $\langle D\Phi_3(0, 0, 0, 0, w_0)[0, 0, 0, l, 0], k \rangle(z) = \sum_{j=1}^n \overline{k_j(z)l_j(z)}$ so

$$\overline{h(z)} \left\langle \frac{D^2\rho(z, f(z))[k(z)]}{\frac{\partial \rho}{\partial \overline{w_1}}(z, f(z))}, k(z) \right\rangle = \sum_{j=1}^n \overline{h(z)k_j(z)l_j(z)}$$

so, taking real parts of both sides and integrating,

$$(5) \quad \int_0^{2\pi} \operatorname{Re} \frac{\overline{h(z)}}{\frac{\partial \rho}{\partial \overline{w_1}}(z, f(z))} D^2\rho(z, f(z))[k(z)][k(z)] \frac{d\theta}{2\pi} = \int_0^{2\pi} \operatorname{Re} \sum_{j=1}^n \overline{h(z)k_j(z)l_j(z)} \frac{d\theta}{2\pi}.$$

The right side of (5) is 0 since $k_j(0) = 0$ for all j . However the integrand on the left side is positive since

$$\frac{\overline{h(z)}}{\frac{\partial \rho}{\partial \overline{w_1}}(z, f(z))}$$

is positive, by definition of h , and since the real Hessian of ρ in w is positive definite on complex tangents on the fibers of K . Thus $k = 0$, so $D\Phi_3(0, 0, 0, 0, w_0)[0, 0, 0, l, 0] = 0$, hence $l = 0$.

In a similar manner, we can show that $D_{(u,v,k,l)}\Phi(0, 0, 0, 0, w_0)$ is a surjective linear map. Let $(u' + \mathbf{R}, m, c)$ be an element in the target space. Then $D\Phi(0, 0, 0, 0, w_0)[u'/I, 0, 0, 0, 0]$ has first coordinate $u' + \mathbf{R}$, so we may assume without loss of generality that $u' = 0$ in $W_R^{1,2}/\mathbf{R}$, but in the domain we must restrict ourselves to the subspace where $u = b/I$ for some real constant b . Similarly, if

$$b = \frac{\operatorname{Re} \langle c, X(0)\overline{g(0)} \rangle}{\left(\frac{1}{I}(0)\right) |X(0)|^2}$$

then $\operatorname{Re}\langle D\Phi_4(0, 0, 0, 0, w_0)[b/I, 0, 0, 0, 0], X(0)\overline{g(0)}\rangle = \operatorname{Re}\langle c, X(0)\overline{g(0)}\rangle$. Hence we may assume that $\operatorname{Re}\langle c, X(0)\overline{g(0)}\rangle = 0$, provided that we restrict the domain to the subspace in the domain where $b = 0$ (i.e., $u = 0$). Next, if we let $v =$ the constant function $\operatorname{Im}\langle c, X(0)\overline{g(0)}\rangle/(|X(0)|^2)$, then

$$\operatorname{Im}\langle D\Phi_4(0, 0, 0, 0, w_0)[0, v, 0, 0, 0], X(0)\overline{g(0)}\rangle = \operatorname{Im}\langle c, X(0)\overline{g(0)}\rangle$$

so we may assume $0 = \langle c, X(0)\overline{g(0)}\rangle = \overline{X(0)}c_1$, so $c_1 = 0$, provided we restrict the domain to the subspace where $v(0) = 0$. Then we must show that $D\Phi_4(0, 0, 0, 0, w_0)$ maps $H^{1,2}(\Delta)^{n-1}$ onto the subspace of \mathbf{C}^n of complex dimension $n - 1$ of n -tuples with first coordinate zero; this is obvious. Thus we now restrict our domain to those $k \in H^{1,2}(\Delta)^{n-1}$ for which $k(0) = \vec{0}$.

We must then show that the image of

$$(6) \quad \{0\} \times \{v \in W_R^{1,2}(\Delta) | v(0) = 0\} \times \{k \in H^{1,2}(\Delta)^{n-1} | k(0) = \vec{0}\} \times H^{1,2}(\Delta)^n$$

under $D_{(u,v,k,l)}\Phi_2(0, 0, 0, 0, w_0) - D_{(u,v,k,l)}\Phi_3(0, 0, 0, 0, w_0)$, is onto $W^{1,2}(\Delta)^n$. First we show the image dense and then show it to be closed. Suppose that $m \in W^{1,2}(\Delta)^n$ is orthogonal to the image. Then clearly $m \in H_0^{1,2}(\Delta)^n$. Further, m is orthogonal to $(1 + v + \bar{v}i, 0, 0, \dots, 0)$ for all $v \in W_R^{1,2}(\Delta)$ with $v(0) = 0$, so orthogonal to all $(p, 0, 0, \dots, 0)$ where $p \in H_0^{1,2}(\Delta)$; thus $0 = m_1$, the first coordinate of m . Then in $W^{1,2}(\Delta)^n$, m is real orthogonal to $D\Phi_2(0, 0, 0, 0, w_0)[0, 0, k, 0, 0]$ for all $k \in H_0^{1,2}(\Delta)^{n-1}$ with $k(0) = \vec{0}$. Let $k = mX$. Then

$$\begin{aligned} (7) \quad 0 &= \langle D\Phi_2(0, 0, 0, 0, w_0)[0, 0, mX, 0, 0], m \rangle_{1,2} \\ &= C \operatorname{Re}\langle D\Phi_2(0, 0, 0, 0, w_0)[0, 0, mX, 0, 0], m \rangle_2 \\ &\quad + \operatorname{Re}\langle \frac{\partial}{\partial \theta}(D\Phi_2(0, 0, 0, 0, w_0)[0, 0, mX, 0, 0]), \frac{\partial}{\partial \theta}m \rangle_2. \end{aligned}$$

Now $D\Phi_2(0, 0, 0, 0, w_0)$ acts pointwise on elements in $H_0^{1,2}(\Delta)^{n-1}$; we may represent its action by

$$k(z) \mapsto C(k)(z)(1, 0, 0, \dots, 0) + \frac{1}{\frac{\partial \rho}{\partial w_1}(z, f(z))} \left(\tilde{k}(z)\Psi(z) \right),$$

where $\Psi(z)$ is the real Hessian of ρ and $\tilde{k}(z) = (\operatorname{Re}k_1(z), \operatorname{Im}k_1(z), \operatorname{Re}k_2(z), \operatorname{Im}k_2(z), \dots, \operatorname{Re}k_n(z), \operatorname{Im}k_n(z))$ is the realification of $k(z)$. We must complexify $\tilde{k}(z)\Psi(z)$ before multiplication by $\frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(z, f(z))}$. Thus (7) equals

$$\begin{aligned}
& C\operatorname{Re}\left\langle \frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))} \widetilde{mX\Psi}, m \right\rangle_2 \\
& + \operatorname{Re}\left\langle \frac{\partial}{\partial \theta} \left(\frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))} (\widetilde{mX\Psi}) \right), \frac{\partial}{\partial \theta} m \right\rangle_2 \\
& \text{(using the fact that the first coordinate of } m \text{ is zero)} \\
& = C\operatorname{Re}\left\langle \widetilde{mX\Psi}, \frac{mX}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))X} \right\rangle_2 \\
& + \operatorname{Re}\left\langle \frac{\partial}{\partial \theta} \left(\frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))} (\widetilde{mX\Psi}) \right), \frac{\partial}{\partial \theta} (mX X^{-1}) \right\rangle_2 \\
& = C\operatorname{Re}\left\langle \widetilde{mX\Psi}, \frac{mX}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))X} \right\rangle_2 \\
& + \operatorname{Re}\left\langle \frac{\partial}{\partial \theta} \left(\frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))} (\widetilde{mX\Psi}) \right) + \left(\frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))} \right) ((\widetilde{mX})_\theta \Psi) \right. \\
& \left. + \left(\frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))} \right) (\widetilde{mX\Psi}_\theta), \frac{\partial}{\partial \theta} (mX) X^{-1} + (mX) \frac{\partial}{\partial \theta} X^{-1} \right\rangle_2 \\
& = C\operatorname{Re}\left\langle \widetilde{mX\Psi}, \frac{mX}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))X} \right\rangle_2 \\
& + \operatorname{Re}\left\langle \left(\frac{1}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))} \right) ((\widetilde{mX})_\theta \Psi), \frac{\partial}{\partial \theta} (mX) X^{-1} \right\rangle_2 \\
& + \text{other terms} \\
(8) \quad & = C\operatorname{Re}\left\langle \widetilde{mX\Psi}, \frac{mX}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))X} \right\rangle_2 \\
& + \operatorname{Re}\left\langle ((\widetilde{mX})_\theta \Psi), \frac{(mX)_\theta}{\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))X} \right\rangle_2 \\
& + \text{other terms.}
\end{aligned}$$

We recall that $\frac{\partial \rho}{\partial \bar{w}_1}(\cdot, f(\cdot))X$ must be a real-valued function. Since $\Psi(z)$ is positive definite on the complex tangent space to K_z at $(z, f(z))$, the first two terms in this last sum are greater than or equal to $\mathcal{K}'(C\|mX\|_2^2 + \|(mX)_\theta\|_2^2) \geq \mathcal{K}''(C\|m\|_2^2 + \|m_\theta\|_2^2)$. (Recall that the real inner product of a and b is $\operatorname{Re}\langle a, b \rangle$. See also the

argument leading up to (5).) The last group of terms can be written in modulus as less than or equal to $C_2\|m\|_2\|m_\theta\|_2 \leq C_3\|m\|_2^2/\epsilon^2 + C_3\|m_\theta\|_2^2\epsilon^2$, where C_3 doesn't depend on m . Choose ϵ so small that $C_3\epsilon^2 < \frac{1}{2}\mathcal{K}''$ and then assume that C was chosen large enough that the last equality of (8) is greater than or equal to a constant times $\mathcal{K}/2(C_1\|m\|_2^2 + \|m_\theta\|_2^2) \geq C'\|m\|_{1,2}^2$. We may then conclude that $m = 0$. Hence in $(W^{1,2}(\Gamma))^n$, the image of (6) under $D_{(u,v,k,l)}\Phi_2(0,0,0,0,w_0) - D_{(u,v,k,l)}\Phi_3(0,0,0,0,w_0)$ is dense.

We need to show that the image of (6) under $D_{(u,v,k,l)}(\Phi_2 - \Phi_3)(0,0,0,0,w_0)$ is closed. Consider a convergent $\{m^j\} \in W^{1,2}(\Delta)^n$ in its image; then there exist $\{v^j\}$, $\{k^j\}$ and $\{l^j\}$ with $k^j(0) = \vec{0}$ and $v^j(0) = 0$ such that

$$m^j = D\Phi(0,0,0,0,w_0)[0, v^j, k^j, l^j, 0].$$

Then

$$m^a - m^b = (1 + v^a - v^b + \tilde{v}^a i - \tilde{v}^b i)\bar{g} + D\Phi_2(0,0,0,0,w_0)[0,0,k^a - k^b,0,0] + \bar{l}^a - \bar{l}^b$$

so, using (4),

$$\langle m^a - m^b, f \rangle = 1 + v^a - v^b + \tilde{v}^a i - \tilde{v}^b i + \langle \bar{l}^a - \bar{l}^b, f \rangle$$

converges in $W^{1,2}(\Delta)$. Projecting the right side to $H^{1,2}$, we conclude that $\{v^a + \tilde{v}^a i\}$ converges, so $\{v^a\}$ does as well in $W^{1,2}(\Delta)$. Thus we may assume without loss of generality that $v^j = 0$. Thus

$$m^a - m^b = D\Phi_2(0,0,0,0,w_0)[0,0,k^a - k^b,0,0] + \bar{l}^a - \bar{l}^b,$$

so

$$\langle m^a - m^b, (k^a - k^b)/X \rangle_{1,2} = \langle D\Phi_2(0,0,0,0,w_0)[0,0,k^a - k^b,0,0], (k^a - k^b)/X \rangle_{1,2}.$$

By reasoning above in (8), the right hand side is $\geq C_4\|k^a - k^b\|_{1,2}^2$ and the left hand side is $\leq C_5\|m^a - m^b\|_{1,2}\|k^a - k^b\|_{1,2}$. Hence $\|k^a - k^b\|_{1,2} \leq C_6\|m^a - m^b\|_{1,2}$. We

conclude that $\{k^a\}$ converges in $H^{1,2}(\Delta)$ to some k , and similarly that $\{l^a\}$ does. This proves closure of the image.

From the conclusions of the last several paragraphs, we conclude that the partial derivative $D_{(u,v,k,l)}\Phi(0,0,0,0,w_0)$ is surjective.

We conclude by the implicit function theorem that there exist an $N(w_0)$ and unique mappings $u, v, k, l : N(w_0) \rightarrow (W_R^{1,2}(\Gamma) \times W_R^{1,2}(\Delta) \times H^{1,2}(\Delta)^{n-1} \times H^{1,2}(\Delta))^n$ such that for $w \in N(w_0)$,

$$\Phi_1(u(w), v(w), k(w), l(w), w) = \mathbf{0} + \mathbf{R},$$

$$\Phi_3(u(w), v(w), k(w), l(w), w) - (1 + v(w) + \widetilde{v(w)i})\Phi_2(u(w), v(w), k(w), l(w), w) = \mathbf{0},$$

and

$$\Phi_4(u(w), v(w), k(w), l(w), w) = w$$

$$\text{so } (1 + v(w) + \widetilde{v(w)i})\Phi_2(u(w), v(w), k(w), l(w), w) = \Phi_3(u(w), v(w), k(w), l(w), w).$$

We also find functions $F(w) \equiv F(u(w), v(w), k(w))$ and $\tilde{G}(w) \equiv G(l(w))$ in the same small neighborhood $N(w_0)$, finding that $F(w)(0) = w$; for fixed $w \in N(w_0)$,

$$\rho(z, F(w)(z))$$

is constant and for fixed w

$$\begin{aligned} & \frac{(1 + v(w)(z) + \widetilde{v(w)}(z)i) \left(\frac{\partial \rho}{\partial w_1}(z, F(w)(z)), \frac{\partial \rho}{\partial w_2}(z, F(w)(z)), \dots, \frac{\partial \rho}{\partial w_n}(z, F(w)(z)) \right)}{\sum_{j=1}^n (\overline{F_j(w)}(z)) \left(\frac{\partial \rho}{\partial w_j}(z, F(w)(z)) \right)} \\ &= (\overline{\tilde{G}_1(w)}(z), \overline{\tilde{G}_2(w)}(z), \dots, \overline{\tilde{G}_n(w)}(z)), \end{aligned}$$

so

$$\begin{aligned} & \frac{(1 + v(w)(z) - \widetilde{v(w)}(z)i) \left(\frac{\partial \rho}{\partial w_1}(z, F(w)(z)), \frac{\partial \rho}{\partial w_2}(z, F(w)(z)), \dots, \frac{\partial \rho}{\partial w_n}(z, F(w)(z)) \right)}{\sum_{j=1}^n (F_j(w)(z)) \left(\frac{\partial \rho}{\partial w_j}(z, F(w)(z)) \right)} \\ &= (\tilde{G}_1(w)(z), \tilde{G}_2(w)(z), \dots, \tilde{G}_n(w)(z)), \end{aligned}$$

which means that the complex tangent space to $K_z^{\rho(z, F(w)(z))}$ at $F(w)(z)$ is

$$\{(w_1, w_2, \dots, w_n) \mid \sum_{j=1}^n \tilde{G}_j(w)(z)w_j = 1 + v(w) - \widetilde{v(w)}i\}$$

and

$$\sum_{j=1}^n F_j(w)\tilde{G}_j(w) = 1 + v(w) - \widetilde{v(w)}i.$$

Since the left hand side is analytic and the right hand side is conjugate analytic, the right side is constant (for fixed w); this constant is 1 for $w = w_0$ so is nonzero for w near w_0 . Letting $G(w) = \tilde{G}(w)/(1 + v(w) - \widetilde{v(w)}i)$ for w near w_0 , we have F, G satisfying the requirements of the theorem. This concludes the proof of Theorem 1. \square

§2 Extension of the implicit functions F and G .

For $t > 0$, we let L^t be the compact set which is the image of K^t under the mapping

$$\Gamma \times \mathbf{C}^n \longrightarrow \Gamma \times \mathbf{C}^{2n}$$

$$(z, w_1, w_2, \dots, w_n) \longmapsto (z, w_1, w_2, \dots, w_n, I_1(z, w), I_2(z, w), \dots, I_n(z, w)),$$

where $w = (w_1, w_2, \dots, w_n)$ and $(I_1(z, w), I_2(z, w), \dots, I_n(z, w)) =$

$$\left(\frac{\frac{\partial \rho}{\partial w_1}(z, w)}{\sum_{j=1}^n w_j \frac{\partial \rho}{\partial w_j}(z, w)}, \frac{\frac{\partial \rho}{\partial w_2}(z, w)}{\sum_{j=1}^n w_j \frac{\partial \rho}{\partial w_j}(z, w)}, \dots, \frac{\frac{\partial \rho}{\partial w_n}(z, w)}{\sum_{j=1}^n w_j \frac{\partial \rho}{\partial w_j}(z, w)} \right).$$

Then L^t is a C^5 -smooth manifold embedded in \mathbf{C}^{2n+1} , as it is a C^5 -smooth graph over K^t . We shall show that L^t is totally real and that the accumulation points over Γ of the analytic disks parametrized by

$$\text{int } \Delta \longrightarrow \mathbf{C}^{2n+1}$$

$$z \longmapsto (z, F(w)(z), G(w)(z))$$

lie in L^t . Results of Čirka [Či] will then show that $F(w)$ and $G(w)$ are in $C^4(\Gamma)$ for $w \in N(w_0)$.

The following lemma is closely related to a lemma in [Wb].

Lemma 1. *L^t is totally real.*

Proof. Under the projection map

$$(z, w_1, w_2, \dots, w_n, v_1, v_2, \dots, v_n) \rightarrow (z, w_1, w_2, \dots, w_n)$$

complex tangents are carried to complex tangents, so a complex tangent to L^t at $(z, w_1, w_2, \dots, w_n, v_1, v_2, \dots, v_n)$ must be of the form

$$(9) \quad (0, \mathcal{T}, \mathcal{U})$$

where \mathcal{T} is a complex tangent to K_z^t at (w_1, w_2, \dots, w_n) . Then for (9) to be a complex tangent, it will suffice that $\frac{\partial I_j}{\partial \mathcal{T}} = 0$ for $j = 1, 2, \dots, n$. Suppose, indeed, that $\mathcal{T} = (a_1, a_2, \dots, a_n)$ is a complex tangent and we map $\lambda \mapsto (w_1, w_2, \dots, w_n) + \lambda(a_1, a_2, \dots, a_n)$. Then $\sum_{j=1}^n a_j \frac{\partial \rho}{\partial w_j} = 0$, and for (9) to be a complex tangent it will suffice that $\frac{\partial I_j}{\partial \lambda} = 0$ for all j . We write $I(z, w) = (I_1(z, w), I_2(z, w), \dots, I_n(z, w)) = \frac{1}{S(z, w)} (\frac{\partial \rho}{\partial w_1}(z, w), \frac{\partial \rho}{\partial w_2}(z, w), \dots, \frac{\partial \rho}{\partial w_n}(z, w))$. Then

$$(10) \quad \begin{aligned} \frac{\partial I}{\partial \lambda} &= \frac{\partial(\frac{1}{S})}{\partial \lambda} (\frac{\partial \rho}{\partial w_1}(z, w), \frac{\partial \rho}{\partial w_2}(z, w), \dots, \frac{\partial \rho}{\partial w_n}(z, w)) \\ &+ \frac{1}{S} (\sum_{j=1}^n \bar{a}_j \frac{\partial^2 \rho}{\partial w_1 \partial \bar{w}_j}(z, w), \sum_{j=1}^n \bar{a}_j \frac{\partial^2 \rho}{\partial w_2 \partial \bar{w}_j}(z, w), \dots, \sum_{j=1}^n \bar{a}_j \frac{\partial^2 \rho}{\partial w_n \partial \bar{w}_j}(z, w)). \end{aligned}$$

If we take the complex inner product of (10) with $(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n)$ (the latter on the right), then the first term drops out since $\sum_{j=1}^n a_j \frac{\partial \rho}{\partial w_j} = 0$. The second term becomes

$$\frac{1}{S} \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} a_i \bar{a}_j$$

which (exists and) is nonzero because $S \neq 0$ and K_z^t is strictly pseudoconvex. We conclude that L^t indeed has no nonzero complex tangents. \square

Lemma 2. $(\widehat{L^t})_z = L_z^t$ for $z \in \Gamma$.

Proof. It is easy to show that $(\widehat{L^t})_z = (\widehat{L_z^t})$ so all we must do is show that $(\widehat{L_z^t}) = L_z^t$. Suppose that $(w', v') \in \widehat{L_z^t}$. Then $w' \in \widehat{K_z^t}$ since projection is analytic, so $\rho(z, w') \leq 1$. Now the polynomial $w_1 v_1 + w_2 v_2 + \dots + w_n v_n - 1$ is identically 0 on L^t so is 0 on $\widehat{L_z^t}$. Thus $w' \neq 0$. Let $\mathcal{P}(w_1, w_2, \dots, w_n, v_1, v_2, \dots, v_n) = (w_1, w_2, \dots, w_n)$

and $\mathcal{Q}(w_1, w_2, \dots, w_n, v_1, v_2, \dots, v_n) = (v_1, v_2, \dots, v_n)$. Now for $|w|$ sufficiently small, $T_w \equiv \{(v_1, v_2, \dots, v_n) \mid w_1 v_1 + w_2 v_2 + \dots + w_n v_n = 1\}$ doesn't meet $\widehat{\mathcal{Q}(L_z^t)}$. Let $\mathcal{B}_z = \{(w_1, w_2, \dots, w_n) \mid T_w \cap \widehat{\mathcal{Q}(L_z^t)} = \emptyset\}$. Then \mathcal{B}_z is open. We claim that $\mathcal{B}_z \cap \mathcal{P}(\widehat{L_z^t}) = \emptyset$. Suppose $w' \in \mathcal{P}(\widehat{L_z^t})$, and $(w', v') \in \widehat{L_z^t}$ projects to it. Then $v' \in \mathcal{Q}(\widehat{L_z^t}) \subset \widehat{\mathcal{Q}(L_z^t)}$ and $v' \in T_{w'}$ so $w' \notin \mathcal{B}_z$, so the claim holds. Consider the connected component of $\mathcal{B}_z \cap \{w \mid \rho(z, w) < 1\}$ which contains 0. Construct a continuous path $w(s)$, $0 \leq s \leq 1$, from 0 to a point in the boundary of this component. Then we claim that $T_{w(1)}$ meets $\widehat{\mathcal{Q}(L_z^t)}$ in only one point, which must be in $\mathcal{Q}(L_z^t)$. If they met in a point in $\widehat{\mathcal{Q}(L_z^t)} \setminus \mathcal{Q}(L_z^t)$ then as $s \rightarrow 1$, $T_{w(s)}$ approaches $\widehat{\mathcal{Q}(L_z^t)} \setminus \mathcal{Q}(L_z^t)$, so by the Oka-Weil Theorem, must also approach a point of $\mathcal{Q}(L_z^t)$. (What one must do is consider the reciprocals of the complex affine maps which vanish on $T_{w(s)}$. They are analytic in a neighborhood of $\widehat{\mathcal{Q}(L_z^t)}$, so satisfy the maximum modulus principle with respect to $\mathcal{Q}(L_z^t)$. Clearly then $T_{w(s)}$ cannot approach a point of $\widehat{\mathcal{Q}(L_z^t)} \setminus \mathcal{Q}(L_z^t)$ without approaching a point of $\mathcal{Q}(L_z^t)$.) Hence $T_{w(1)}$ contains a point of $\mathcal{Q}(L_z^t)$, say v' . But by strict hypoconvexity of $\widehat{K_z^t}$, there is only one point w' in $\widehat{K_z^t}$ such that $\sum_{j=1}^n w'_j v'_j = 1$; we must have $w(1) = w'$ and $w' \in K_z^t$. Thus \mathcal{B}_z is the entire interior of $\widehat{K_z^t}$. We conclude that $\mathcal{P}(\widehat{L_z^t})$ is only K_z^t .

Then $\widehat{L_z^t}$ is a compact set in $K_z^t \times \mathbf{C}^n$. Let (w', v') be the point in L_z^t with first coordinate w' . Suppose that $(w', v'') \in \widehat{L_z^t} \setminus L_z^t$. Let Q be a polynomial in v which vanishes at v' but equals 1 at v'' . Choose a sequence of complex $(n-1)$ -dimensional hyperplanes external to K_z^t approaching w' . Suppose they are defined by the vanishing of the complex affine functions $Q_i(w)$. Regarding $1/Q_i(w)$ as a function of both w and v , let M_i be the maximum of $1/Q_i$ on L_z^t . Then for large i , the modulus of $Q(v) \frac{1}{(M_i Q_i(w))^i}$ is larger at (w', v'') than at any point on L_z^t , a contradiction since $Q(v) \frac{1}{(M_i Q_i(w))^i}$ is analytic on a large compact subset of $L_z^t \times \mathbf{C}^n$, hence is approximable by polynomials. This proves $\widehat{L_z^t} = L_z^t$. \square

From Theorem 1, we have a parametrization of analytic graphs with boundaries in the various K^t by their values at zero. We wish to show that this parametrization

is essentially unique. To do this, we need the following lemma.

Lemma 3. *Suppose that $w_0 \in \mathbf{C}^n$, $N(w_0)$ an open neighborhood such that there exist $F(w), G(w)$ with the properties arising from Theorem 1. Then for a smaller neighborhood N' of w_0 the mapping $P : \mathbf{C}^n \rightarrow \mathbf{R}$ given by $P(w) = \rho(z, F(w)(z))$ (where z is any point in Γ) is C^1 in $N'(w_0)$ with nonzero gradient.*

Proof. Smoothness is trivial: P is a composition of C^1 functions. It will suffice to restrict ourselves to the case where $g = (1, 0, 0, 0, \dots, 0)$ in Theorem 1, since, using notation from Theorem 1, $P(w) = \tilde{P}(w \cdot M(0))$, where $\tilde{P}(w) \equiv \tilde{\rho}(z, \tilde{F}(w)(z))$ and $M(0)$ is invertible. Let $P'(u, v, k)(z) = \rho(z, F(u, v, k)(z))$, in the notation of Theorem 1. Now $DP'(0, 0, 0)[u, v, k]$ changes only with changes in u , and is injective in u (the proof is similar to an argument in the proof in Theorem 1 and the argument in section 2 of [F1]; $DP'(0, 0, 0)[u, v, k] = Iu$, where $I(z) = 2\text{Re}(X(z)\frac{\partial \rho}{\partial w_1}(z, f(z)))$.) Thus it suffices to show that the derivative of $u(w)$ in w is not degenerate at w_0 , and for this it suffices to show that the derivative of the mapping $e(w) = u(w)(0)$ at w_0 is nondegenerate. Begin with the equation $w = f(0) + (u(w)(0))X(0)f(0) + v(w)(0)iX(0)f(0) + k(w)(0)$; taking the real inner product of both sides with $X(0)\bar{g}(0)$, we obtain $\text{Re}\langle w, X(0)\bar{g}(0) \rangle = \text{Re}X(0) + \text{Re}(u(w)(0))|X(0)|^2 = \text{Re}X(0) + e(w)|X(0)|^2$. Differentiating with respect to w , we find $\text{Re}\langle w, X(0)\bar{g}(0) \rangle = De(w_0)[w]|X(0)|^2$. Letting $w = X(0)f(0)$ we find that $De(w_0)[X(0)f(0)] = 1 \neq 0$. (This uses the fact that X has winding number zero, so $X(0) \neq 0$.) Thus $De(w_0)$ is non-degenerate, as desired. \square

We henceforth assume that $N(w_0)$ has been shrunk so that it possesses the property found in Lemma 3.

Lemma 4. *Suppose that $N(w_0), F(w), G(w)$ arise out of Theorem 1 and Lemma 3 and ϕ is a function in $H^\infty(\Delta)^n$ such that $\phi(z) = w$ for some $w \in N(w_0)$ and $\rho(z, \phi(z)) \leq P(w)$ for almost every $z \in \Gamma$. Then $F(w) \equiv \phi$.*

Proof. If we have $F(w) \not\equiv \phi$, then the function $\sum_{j=1}^n G_j(w)(F_j(w) - \phi_j)$ is not identically zero, because $\sum_{j=1}^n G_j(w)(z)(F_j(w)(z) - \phi_j(z)) = 1 - \sum_{j=1}^n G_j(w)(z)\phi_j(z)$

which, for a.e. $z \in \Gamma$, can only equal zero at $z \in \Gamma$ if $F(w)(z) = \phi(z)$ by the strict hypoconvexity of ρ and by the fact that $\rho(z, \phi(z)) \leq P(w)$ for a.e. $z \in \Gamma$. However, if $F(w) \not\equiv \phi$ then $F(w)(z) \neq \phi(z)$ on a set of positive measure in Γ . Let $t_0 = P(w) \equiv \rho(z, F(w)(z))$. By Lemma 3, we can choose a sequence $\{w^j\}$, such that $w^j \rightarrow w$, $P(w^j) = t_j$ and $t_j \downarrow t_0$. Consider the function $z \mapsto 1 - \sum_{j=1}^n G_j(w^i)(z)\phi_j(z)$ for $i \geq 1$; we show in the next paragraph that it has no zeroes on the disk (deforming $G(w^i)$ through g^t to a function near 0 so that $\{v \mid \sum_{j=1}^n g^t(z)v_j = 1\}$ doesn't meet $\widehat{K_z^{t_0}}$ for $z \in \Gamma$). However, as $i \rightarrow \infty$, $1 - \sum_{j=1}^n G_j(w^i)\phi_j$ converges uniformly on compact sets to $1 - \sum_{j=1}^n G_j(w)\phi_j$, since G is continuous in w . By Hurwitz' theorem this means that $1 - \sum_{j=1}^n G_j(w)\phi_j$ is identically zero on the disk since $1 - \sum_{j=1}^n G_j(w)(0)\phi(0) = 1 - \sum_{j=1}^n G_j(w)(0)F_j(w)(0) = 0$. (This holds because $F(w)(0) = w$ by Theorem 1.) Thus we conclude that $\sum_{j=1}^n G_j(w)(F_j(w) - \phi_j)$ is identically zero on Δ , so over Γ in particular. By the strict hypoconvexity of the fibers, we conclude that $F(w) = \phi$ identically, as desired.

Now as to the deformation of $G(w^i)$ described, assuming i fixed: we can choose a homotopy $\{f^t\}$ of $F(w^i)$ such that $\rho(z, f^t(z)) \equiv t$ for all $t_i \leq t \leq R$ and $z \in \Gamma$, $|f^t(z)| = t$ for $t \geq R$ and $z \in \Gamma$, and $f^{t_i} = F(w^i)$. Then let $g^t(z) = D_w \rho(z, f^t(z)) / \sum_j f_j^t(z) \frac{\partial \rho}{\partial w_j}(z, f^t(z))$ for $t_i \leq t \leq R$ and $\overline{f^t(z)}/t^2$ if $t > R$. Extend f^t, g^t harmonically to the closed disk. If R' is chosen large enough then $|\sum_{j=1}^n g_j^{R'}(z)\phi_j(z)| \leq \frac{1}{2}$ for $z \in \Delta$. We then claim that for r near 1, there exists an $\epsilon > 0$ such that $|1 - \sum_{j=1}^n g_j^t(z)\phi_j(z)| \geq \epsilon$ for $t_i \leq t \leq R'$ and $r \leq |z| < 1$. If not, there exists a convergent sequence $\{z_k, s_k\}$ such that $|z_k| \rightarrow 1$, $\phi(z_k) \rightarrow v$ and $|1 - \sum_{j=1}^n g_j^{s_k}(z_k)\phi_j(z_k)| \rightarrow 0$. Suppose that $(z_k, s_k) \rightarrow (z, s)$. Then $g^{s_k}(z_k) \rightarrow g^s(z)$ and $v \in \widehat{K_z^{t_0}}$ (since the graph of ϕ is in the polynomial hull of K^{t_0}), so $1 - \sum_{j=1}^n g_j^s(z)v_j = 0$. This is a contradiction because $s > t_0$ and the set $\{w \in \mathbf{C}^n \mid 1 - \sum_{j=1}^n g_j^s(z)w_j = 0\}$ does not meet $\widehat{K_z^{t_0}}$. With the existence of r as claimed, we see that $1 - \sum_{j=1}^n G_j(w^i)\phi_j = 1 - \sum_{j=1}^n g_j^{t_i}(z)\phi_j$ has the same winding number on radius r as $1 - \sum_{j=1}^n g_j^{R'}(z)\phi_j$ which is within $\frac{1}{2}$ of 1 on Δ .

Thus $1 - \sum_{j=1}^n G_j(w^i)\phi_j$ has no zeroes inside the disk of radius r , and none outside by definition of r . \square

Corollary. *If $N(w_0), F^0(w), G^0(w), N(w_1), F^1(w), G^1(w)$ both arise out of Theorem 1 and Lemma 3 such that $N(w_0) \cap N(w_1) \neq \emptyset$ then for all $w \in N(w_0) \cap N(w_1)$, $F^0(w) = F^1(w)$ and $G^0(w) = G^1(w)$.*

Proof. Choose $w \in N(w_0) \cap N(w_1)$. Assume without loss of generality that $\rho(z, F^1(w)(z)) \leq \rho(z, F^0(w)(z))$. Applying Lemma 4, letting $\phi = F^1(w)$ and $F = F^0$, we have $F^0(w) = F^1(w)$, from which $G^0(w) = G^1(w)$ follows immediately. \square

We now explain in rough terms, without stating a precise theorem, how we shall begin with an f, g as in Theorem 1 and construct $F(w), G(w)$ for many w . In our applications, the set where $\rho = R$ will be equal to the set where $|w|$ is some constant R , so we could begin with $f(z) = (R, 0, 0, \dots, 0)$ and $g(z) = (\frac{1}{R}, 0, 0, \dots, 0)$. Using Theorem 1, we can construct a neighborhood N of $f(0)$ and associated $F(w), G(w)$. Suppose v is a boundary point of $N(f(0))$ and suppose $\{w^j\}$ is a sequence in N , $0 < P(w^j) \leq R$, such that $w^j \rightarrow v$ and $P(w^j) \rightarrow t > 0$, with associated $(F(w^j), G(w^j))$. By Corollaries 1 and 2 in section 2 of [HMa], a local uniform limit (ϕ, ψ) of the $(F(w^j), G(w^j))$ has a graph whose accumulation points lie in L^t (see the beginning of §2 for the definition of L^t). Then the functions ϕ, ψ are C^4 -smooth on Δ from [Či] and they satisfy the properties that f, g do in Theorem 1; hence by Theorem 1, they may be parametrized locally smoothly by $w \mapsto (F'(w), G'(w))$ in some $N'(v)$, regarding $F'(w)$ and $G'(w)$ as elements of $H^{1,2}(\Delta)^n$. By the Corollary, where $N'(v)$ meets $N(v)$, $(F', G') = (F, G)$. Thus we can extend F, G as far as the graphs of the limiting functions $(F(v), G(v))$ continue to have boundaries where ρ is C^6 and strictly hypoconvex, and $0 < \rho \leq R$.

If ρ has smoothness C^k for $k > 6$ then the various $F(w)$ and $G(w)$ extend to $C^{k-2}(\Gamma)$, again using Čirka's result, since then L^t is a totally real C^{k-1} manifold.

§3 Polynomial hulls with hypoconvex fibers.

Theorem 2. *Suppose ρ satisfies (3) with $S = 0$, so the fibers of K_z^t of K^t enclose*

the origin in \mathbf{C}^n . Then $\widehat{K} \setminus K$ is the union of graphs of elements of $A(\Delta) \cap C^4(\Gamma)$ (whose boundaries lie in some K^t , $t \leq 1$). Given a point in $\partial\widehat{K} \cap (\text{int } \Delta \times \mathbf{C}^n)$, there is precisely one element of $H^\infty(\Delta)$ whose graph is in $\widehat{K} \setminus K$ and passes through that point. For all $z \in \Delta$, \widehat{K}_z is hypoconvex, with C^1 boundary; in fact, we have $\widehat{K}_0 = \{w \in \mathbf{C}^n \mid P(w) \leq 1\}$.

Proof. We easily find f, g as in Theorem 1; we can take $f(z) = (R, 0, 0, \dots, 0)$ and $g(z) = (\frac{1}{R}, 0, 0, \dots, 0)$. Then we can construct an open U and F, G, P as before. We claim that we can extend F, G and P smoothly to $\{w \mid 0 < |w| \leq R\}$ and P continuously to where $w = 0$. The extension of F, G to the set where $|w| = R$ is obvious, and then the extension to a neighborhood of this sphere is by application of Theorem 1 and Lemmas 1-4. Then $P(w) = R$ if $|w| = R$, so $D_w P(w)$ is a real multiple of w (it is nonzero, by Lemma 3); we claim it is a positive multiple. Were it negative, then for some $|w| > R$, we would have $|F(w)| \leq R$ on Γ , but $|F(w)(0)| > R$, in violation of the maximum modulus principle. Suppose U is the maximal open subset of $\{w \mid 0 < |w| < R\}$ to which F and G (so P also) can be extended C^1 -smoothly. If U excludes points in the annulus $0 < |w| \leq R$, then take an open segment in U one of whose vertices w is not in U , $0 < |w| < R$ and the other of which lies on $\{w \mid |w| = R\}$. We claim that P is bounded on the segment. If not, then at some point w' on the segment $P(w') = R$, so $F(w')$ and $G(w')$ are constant functions of modulus $R, \frac{1}{R}$, respectively, so $|w| = R$, a contradiction. (Recall that $\overline{G(w')(z)}$ is a complex multiple of a normal to the sphere of radius R , in fact, it equals $\frac{1}{R^2} F(w')(z)$; then $1 = \langle F(w'), \overline{G(w')} \rangle = \frac{1}{R^2} |F(w')|^2$ is bounded analytic and real on Γ . Thus $F(w')$ has constant modulus on Δ , so its components are constants, from which we can make the above conclusions.) Then take a sequence $\{w^j\}$ on the segment converging to w , such that $\{P(w^j)\}$ converges. We cannot have $\lim_{j \rightarrow \infty} P(w^j) = 0$ because then a subsequence of $\{F(w_j)\}$ converges locally uniformly to zero, which is impossible because $w^j = F(w^j)(0)$ converges to w which is not zero. Then we can define $F(w), G(w)$ as the local uniform limits

of subsequences of $\{F(w_j)\}$ and $\{G(w_j)\}$. As outlined at the end of §2, $F(w)$ and $G(w)$ are C^4 functions on Γ . Then using Theorem 1, we extend smoothly to a neighborhood of w in such a way as to coincide with F and G on U , a contradiction. Thus F, G extend to be C^1 on the annulus $0 < |w| \leq R$, where F, G must be the natural constants when $|w| = R$: $F(w)(z) = w$ constantly and $G(w)(z) = \bar{w}/|w|^2$.

Suppose $\{w^j\}$ is a sequence which converges to the origin $\vec{0}$ but $t \equiv \lim P(w^j) \neq 0$. Then since $t > 0$, we could define $F(\vec{0})$ and $G(\vec{0})$ to be local uniform limits of $\{F(w^j)\}$ and $\{G(w^j)\}$. However, since $F(\vec{0})(0) = \vec{0}$ we cannot possibly have $\sum_{j=1}^n F_j(\vec{0})(0)G_j(\vec{0})(0)$ equal to 1. Thus $\lim P(w^j) = 0$, which means we can extend P continuously to $\{w \mid |w| \leq R\}$ by defining $P(\vec{0}) = 0$. We also find from this limit that as $w^j \rightarrow 0$, $F(w^j)$ converges uniformly to zero, so for convenience we define $F(\vec{0}) = \vec{0}$.

Let s be the maximum of P on \widehat{K}_0 . (Note that the domain of P clearly contains $\widehat{K}_0 \subset \{w \in \mathbf{C}^n \mid |w| \leq R\}$.) Clearly $s < R$. Suppose $s > 1$. Let v be a point on \widehat{K}_0 where this maximum is attained. Then v is not in the interior of \widehat{K}_0 because P does not attain local maxima where it is smooth, since P has nonzero gradient. The only place P is not smooth when $|w| < R$ occurs when $w = 0$ and P doesn't attain a local maximum there since $P(\vec{0}) = 0$. Thus we can choose a continuous path $v(t)$, $s < t < R$, outside of \widehat{K}_0 such that $P(v(t)) = t$ and as $t \rightarrow s^+$, $v(t) \rightarrow v$.

Then consider

$$(11) \quad \{(z, w) \in \Delta \times \mathbf{C}^n \mid \sum_{j=1}^n G_j(v(t))w_j = 1\}$$

Let $s' =$ the supremum of all t such that (11) meets \widehat{K} . Clearly $1 < s \leq s' < R$.

The function

$$M_t(z, w) = \frac{1}{\sum_{j=1}^n G_j(v(t))w_j - 1}$$

on $\Delta \times \mathbf{C}$ is defined on \widehat{K} for $t > s'$. Since $G(v(t)) \in A(\Delta)$ for all t , $M_t(z, w)$ is uniformly approximable on \widehat{K} for $t > s'$ by functions analytic in a neighborhood of

\widehat{K} , hence uniformly approximable by polynomials in a neighborhood of \widehat{K} by the Oka-Weil Theorem. This means that

$$\sup_{(z,w) \in \widehat{K}} |M_t(z,w)| \leq \sup_{(z,w) \in K} |M_t(z,w)|,$$

for $t > s'$. As $t \downarrow s'$, $\sup_{(z,w) \in \widehat{K}} |M_t(z,w)| \rightarrow \infty$ by the definition of s' . However since $s' > 1$, the distance between K^t and K^1 is bounded away from zero uniformly in t , and the singularity set of M_t is no closer to $K^1 = K$ than points in K^t , by strict hypoconvexity of the fibers, so

$$\sup_{(z,w) \in K} |M_t(z,w)|$$

is bounded uniformly in t . This contradicts the previous assertion. Thus $s = 1$ and we find that $\widehat{K}_0 \subset \{w \in \mathbf{C}^n \mid |w| < R, P(w) \leq 1\}$. Thus every point in \widehat{K}_0 is on the graph of some $F(w)$ for which $P(w) \leq 1$. The same holds for all other \widehat{K}_z , $z \in \text{int } \Delta$ by applying a Möbius transformation to the disk sending z to 0 and applying the same argument. Thus \widehat{K} is indeed the union of graphs of elements of $A(\Delta)$ which extend to $C^4(\Gamma)$.

Now given a point in $\partial \widehat{K} \cap (\text{int } \Delta \times \mathbf{C}^n)$, suppose by applying a Möbius transformation that it has the form $(0, w)$. Then $P(w) = 1$, for if $P(w) < 1$, the graphs of $F(w) + e$ are in \widehat{K} for constant analytic vector valued functions e of sufficiently small modulus, so $(0, F(w)(0))$ is in the interior of \widehat{K} . Since $P(w) = 1$, applying Lemma 4, there is no $\phi \in H^\infty(\Delta)^n$ other than $F(w)$ such that $\phi(0) = w$ whose graph is contained in $\widehat{K} \setminus K$, as $\phi(z) \in \widehat{K}_z$ for almost every $z \in \Gamma$ for such a ϕ .

The last statement of the theorem is already known for $z \in \Gamma$. If $z \in \text{int } \Delta$, we may assume without loss of generality that $z = 0$ by applying a Möbius transformation. If $P(w') \leq 1$, then $w' \in \widehat{K}_0$; the converse was shown above. Thus $\partial \widehat{K}_0 = \{w \in \mathbf{C}^n \mid P(w) = 1\}$ and \widehat{K}_0 has C^1 boundary since P is C^1 for $0 < |w| \leq R$. To show that \widehat{K}_0 is hypoconvex: suppose that point $w' \notin \widehat{K}_0$. If $|w'| \leq R$ then the complex affine hyperspace $\{w \in \mathbf{C}^n \mid \langle G(w')(0), \bar{w} \rangle = 1\}$ is external to \widehat{K}_0 (since

$P(w') > 1$) and passes through w' . If $|w'| > R$, it is easy to find such a hyperspace. Thus \widehat{K}_0 is hypoconvex. \square

Theorem 2 will hold if in (3), S has polynomial coordinates, so the sets K_z^t , instead of enclosing the origin, encircle points $S(z)$.

§4 The H^∞ control problem.

Assume that ρ satisfies (3), let γ_ρ be as defined in (2) and let δ_ρ be defined by

$$\delta_\rho \equiv \inf_{f \in A(\Delta)^n} \sup_{z \in \Gamma} \rho(z, f(z)).$$

We assume that S is not analytic, so that $0 < \gamma_\rho \leq \delta_\rho$. For $m = 1, 2, 3, \dots$, let $\rho_m(z, w_1, w_2, \dots, w_n) = \rho(z, \frac{w_1}{z^m}, \frac{w_2}{z^m}, \dots, \frac{w_n}{z^m})$ and define $z^m K^t$ to be the set in $\Gamma \times \mathbf{C}^n$ where ρ_m equals t . Now since $\rho_m(z, z^m g(z)) = \rho(z, g(z))$ for all $g \in H^\infty(\Delta)^n$, we find that $\delta_{\rho_m} \leq \delta_\rho$ and $\gamma_{\rho_m} \leq \gamma_\rho$. We claim that there exists an $m > 0$ such that $\delta_{\rho_m} < \delta_\rho$. The reason for this is as follows: choose any continuous selector α for $\{(z, w) \in \Gamma \times \mathbf{C}^n \mid \rho(z, w) \leq \delta_\rho - \epsilon\}$, where of course ϵ is chosen small enough so that this is possible. Then there exists an n -tuple of harmonic polynomials $q = (q_1, q_2, \dots, q_n)$ such that q is so close to α uniformly, that q is a harmonic polynomial selector for $\{(z, w) \in \Gamma \times \mathbf{C}^n \mid \rho(z, w) \leq \delta_\rho - \frac{\epsilon}{2}\}$. There exists an $m > 0$ such that $z^m q(z)$ has coordinates which are analytic polynomials. Hence $\delta_{\rho_m} \leq \delta_\rho - \frac{\epsilon}{2}$. Let m in fact be the least positive integer such that $\delta_{\rho_m} < \delta_\rho$. Then there exists an n -tuple of harmonic polynomials $q(z)$ such that $\rho(z, q(z)) < \delta_\rho$ and $z^m q(z)$ is analytic. Note that this means $m \geq 1$.

We shall not assume that the level sets of ρ enclose the origin as we did earlier. The role of the zero graph is here replaced by the graph of q .

Theorem 3. *Suppose that ρ satisfies (3) with $S(z) = q(z)$. Then there exists a unique $\phi \in H^\infty(\Delta)^n$ for which $\rho(z, \phi(z)) \leq \gamma_\rho$, for which in fact ϕ extends to be C^4 on Γ and $\rho(z, \phi(z)) = \gamma_\rho$ for all $z \in \Gamma$. Also, if $\rho(\bar{z}, \bar{w}) = \rho(z, w)$, then ϕ is \mathbf{R}^n -valued on the real axis, i.e. $\overline{\phi(\bar{z})} = \phi(z)$. If ρ is C^k where $\rho \neq 0$ for $k > 6$ then ϕ extends to $C^{k-2}(\Gamma)$.*

The condition of being real on the real axis has applications in engineering. See [HMe2]. The condition that $\rho(z, \phi(z)) = \gamma_\rho$ for all z is known as the “frequency domain Bang-Bang principle,” or a condition of “flat performance”. The existence of an H^∞ solution in the theorem has already been proven in [HMa]. If the H^∞ solution is smooth on Γ , then it is the only smooth solution; see [V].

Proof. From the definition of ρ it is clear that $\gamma_\rho \leq R$ and $\delta_\rho \leq R$.

Let us suppose that $m = 1$, i.e., $\delta_{\rho_1} < \delta_\rho$. Applying the technique of Theorem 2 to $\rho_1(w)$ (replacing the zero function by $p(z) = zq(z)$), we obtain $F(w)$ and $P(w)$ to be defined and C^1 in the region where $|w| \leq R + \epsilon$ except when $w = p(0)$. Note that $p(0) \neq \vec{0}$ since q is not analytic. Thus P and F are defined at the origin.

We claim that $\phi \equiv \frac{1}{z}F(\vec{0})$ will solve the H^∞ control problem (2) for ρ and $\gamma_\rho = P(\vec{0})$. Clearly $\phi \in A(\Delta) \cap C^4(\Gamma)$ because $F(\vec{0}) = \vec{0}$ and $F(\vec{0}) \in A(\Delta) \cap C^4(\Gamma)$. Also, if ρ is in fact C^k for $k > 6$ then from the observation at the end of §2, we also find that $\phi \in C^{k-2}(\Gamma)$. The boundary values of ϕ are clearly in $K^{P(\vec{0})}$ since the same holds for $F(\vec{0})$ with respect to $zK^{P(\vec{0})}$. We can show that there is no other element ψ in $H^\infty(\Delta)$ for which $\text{ess sup}_{z \in \Gamma} \rho(z, \psi(z)) \leq P(\vec{0})$ because then there would exist another element $z\psi$ in $H^\infty(\Delta)$ with value $\vec{0}$ at $z = 0$ for which $\rho_1(z, z\psi(z)) \leq \gamma_\rho$ for almost every $z \in \Gamma$. This contradicts Lemma 4.

To prove that $\phi(z) = \overline{\phi(\bar{z})}$ if $\rho(z, w) = \rho(\bar{z}, \bar{w})$, we imitate [HMa]: note that $\overline{\phi(\bar{z})}$ is analytic in z and $\rho(z, \overline{\phi(\bar{z})}) = \rho(\bar{z}, \phi(\bar{z})) = \gamma_\rho$ for all $z \in \Gamma$. Thus $\overline{\phi(\bar{z})}$ is another solution to the H^∞ control problem, so must be the same as $\phi(z)$ by the uniqueness just proven.

Now let us assume that $m > 1$. Applying the work above to ρ_{m-1} , we find there is a unique solution k to the H^∞ control problem for ρ_{m-1} which also happens to be in $A(\Delta) \cap C^4(\Gamma)$. Now $\gamma_\rho \leq \delta_\rho = \delta_{\rho_{m-1}} = \gamma_{\rho_{m-1}}$. (The first equality is by definition of m ; the second because we showed the H^∞ solution for ρ_{m-1} to be in $A(\Delta)$.) However, we already know $\gamma_{\rho_{m-1}} \leq \gamma_\rho$, so $\gamma_{\rho_{m-1}} = \gamma_\rho$. Thus any solution f to the H^∞ control problem for ρ must satisfy $z^{m-1}f(z) = k(z)$ for all $z \in \text{int } \Delta$. Using the known existence of a solution to (2) from [HMa], this proves uniqueness, smoothness of the solution and flatness of performance for the H^∞ control problem for ρ . The property of being \mathbf{R}^n -valued on the real axis follows as well since z^n is real on the real axis. \square

Using a version of Theorem 1, it is easy for us to show that if ρ varies smoothly

then the solution to the H^∞ control problem and the optimal control also vary smoothly.

We suppose that $\rho : \Gamma \times \mathbf{C}^n \times I$ and $S(z, \tau)$ are C^6 where I is an open interval in \mathbf{R} and for every $\tau \in I$, $\rho^\tau(z, w) \equiv \rho(z, w, \tau)$ satisfies (3) with respect to $S^\tau(z) \equiv S(z, \tau)$.

Theorem 4. *Suppose that there exists an $m \geq 1$ which for all $\tau \in I$ is the least positive integer such that $\delta_{\rho_m^\tau} < \delta_{\rho^\tau}$ and suppose that for all $\tau \in I$, $z^m S^\tau(z)$ is in $A(\Delta)$ but $z^{m-1} S^\tau(z)$ is not. If $H(\tau)$ denotes the solution to the H^∞ control problem for ρ^τ , then $H : I \rightarrow H^{1,2}(\Delta)$ and γ_{ρ^τ} are C^1 functions of τ .*

Proof. Fix any point in I , say 0 without loss of generality. We consider the function $\rho_m(z, w, \tau) \equiv \rho(z, \frac{w}{z^m}, \tau)$ on $\Gamma \times \mathbf{C}^n \times I$. We can use reasoning similar to Theorem 1 and Lemmas 1-4, to conclude the existence of C^1 function $F(w, \tau)$ in $N(0) \times I$ such that, following Theorem 3, $\frac{1}{z^m} F(\vec{0}, \tau)$ is the solution to the H^∞ control problem for ρ^τ . (The smoothness of the associated Φ follows from the Lemma in the Appendix, replacing n by $n+1$ and regarding τ as a constant function f_{n+1} .) Then it is easy to see that $H(\tau) = \frac{1}{z^m} F(\vec{0}, \tau)$ and $\gamma_{\rho^\tau} = \rho(1, H(\tau)(1), \tau)$ are C^1 functions of τ . \square

§5 Appendix.

As promised, we prove that Φ is a C^1 -differentiable map in a neighborhood of $(0, 0, 0, 0, w_0)$. Since harmonic conjugation is continuous linear on $W_R^{1,2}(\Gamma)$ (so smooth), $\tilde{u} \in W_R^{1,2}(\Gamma)$ if u is. The product and chain rule for Sobolev functions show that $\Phi_1(u, v, k, l, w)$ is in $W^{1,2}(\Gamma)^n$. We may similarly conclude that $\Phi_2(u, v, k, l, w) \in W^{1,2}(\Gamma)^n$, where the only additional facts that we need is that the denominator is bounded away from 0 for small u, k ; this holds because if u, k are small in $W^{1,2}$ then they are uniformly near 0, so that denominator is uniformly near $\sum_{j=1}^n \overline{f_j(z)} \frac{\partial \rho}{\partial \bar{w}_j}(z, f(z))$, which is never zero for $z \in \Gamma$. Then using the fact that $W^{1,2}$ is an algebra, we conclude that indeed $\Phi_2(u, v, k, l, w) \in W^{1,2}(\Gamma)^n$. If the integrand of Φ_4 is C^1 , so is Φ_4 since the integral is continuous linear.

It will then suffice to prove the following lemma.

Lemma. *Let p be a C^3 function on $\Gamma \times \mathbf{C}^n$. If $P : (W^{1,2}(\Gamma))^n \rightarrow W^{1,2}(\Gamma)$ is given by $P(f)(z) \equiv p(z, f(z))$, then P is a C^1 function.*

Proof. We claim that $DP(f)$ is given by the map $T_f \in L((W_C^{1,2})^n, W_C^{1,2})$ such that $T_f[h](z) = Dp(z, f(z))[0, h(z)]$. We must first check that T_f is in the desired space. This is not difficult: if $h = (h_1, h_2, \dots, h_n)$, then $T_f[h] = \sum_{j=1}^n r_j h_j + s_j \overline{h_j}$, where $r_j(z) = \frac{\partial p}{\partial w_j}(z, f(z))$, $s_j(z) = \frac{\partial p}{\partial \overline{w_j}}(z, f(z))$ are $W^{1,2}$ functions. It is then a simple exercise to show that $\|T_f[h]\|_{1,2}$ is less than or equal to a constant times $\|h\|_{1,2}$. It is also a simple matter to show that T_f varies continuously in f . Next, by Taylor's formula, $P(f+h)(z) - P(f)(z) = p(z, f(z) + h(z)) - p(z, f(z)) = Dp(z, f(z))[0, h(z)] + R(f, h)(z)$, where

$$(12) \quad R(f, h)(e^{i\theta}) = \int_0^1 (1-t) D^2 p(e^{i\theta}, f(e^{i\theta}) + th(e^{i\theta}))[0, h(e^{i\theta})][0, h(e^{i\theta})] dt.$$

For h with small norm in $W^{1,2}$, h also has small supremum norm $\leq C\|h\|_{1,2}$; Then $D^2 p(e^{i\theta}, f(e^{i\theta}) + th(e^{i\theta}))$ is uniformly bounded in θ, t for such h , so the above integral has absolute value bounded by a constant times $\|h\|_\infty^2$. Thus $\|R(f, h)\|_2 \leq C(f)\|h\|_\infty^2 \leq C_1(f)\|h\|_{1,2}^2$, so

$$(13) \quad \lim_{h \rightarrow 0 \text{ in } W^{1,2}} \frac{\|R(f, h)\|_2}{\|h\|_{1,2}} = 0.$$

We may differentiate under the integral sign in (12) to get an integrand so that $|R(f, h)_\theta(e^{i\theta})|$ is bounded above by a constant times $|f_\theta|\|h\|_\infty^2 + \|h\|_\infty^2|h_\theta| + \|h\|_\infty|h_\theta| = \|h\|_\infty(|f_\theta|\|h\|_\infty + \|h\|_\infty|h_\theta| + |h_\theta|)$ which is less than or equal to a constant times $\|h\|_{1,2}(|f_\theta|\|h\|_\infty + \|h\|_\infty|h_\theta| + |h_\theta|)$. Then $\|R(f, h)_\theta\|_2 \leq C_2\|h\|_{1,2}(\|f_\theta\|\|h\|_\infty + \|h\|_\infty|h_\theta| + |h_\theta|)$ and we conclude that

$$\lim_{h \rightarrow 0 \text{ in } W^{1,2}} \frac{\|R(f, h)_\theta\|_2}{\|h\|_{1,2}} = 0,$$

since f_θ is in $L_C^2(\Gamma)$. Combining this with (13), we conclude that P is differentiable.

We showed above that $DP(f)$ is continuous in f , so we are done. \square

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