POLYNOMIAL HULLS AND AN OPTIMIZATION PROBLEM

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Abstract. We say that a subset of $\mathbf{C}^n$ is hypoconvex if its complement is the union of complex hyperplanes. We say it is strictly hypoconvex if it is smoothly bounded hypoconvex and at every point of the boundary the real Hessian of its defining function is positive definite on the complex tangent space at that point. Let B_n be the open unit ball in $\mathbf{C}^n$. Suppose K is a C^∞ compact manifold in $\partial B_1 \times \mathbf{C}^n$, $n > 1$, diffeomorphic to $\partial B_1 \times \partial B_n$, each of whose fibers K_z over ∂B_1 bounds a strictly hypoconvex connected open set. Let $\widehat{K}$ be the polynomial hull of K. Then we show that $\widehat{K} \setminus K$ is the union of graphs of analytic vector valued functions on B_1. This result shows that an unnatural assumption regarding the deformability of K in an earlier version of this result is unnecessary. Next, we study an H^∞ optimization problem. If ρ is a C^∞ real-valued function on $\partial B_1 \times \mathbf{C}^n$, we show that the infimum $\gamma_\rho = \inf_{f \in H^\infty(B_1)^n} \|\rho(z, f(z))\|_\infty$ is attained by a unique bounded f provided that the set $\{(z,w) \in \partial B_1 \times \mathbf{C}^n | \rho(z,w) \leq \gamma_\rho\}$ has bounded connected strictly hypoconvex fibers over the circle.

§1 Introduction and results.

The purpose of this work is to strengthen the results of [Wh] and [V] regarding the existence of analytic structure in a class of polynomial hulls and the solution of an H^∞ optimization problem.

For $w \in \mathbf{C}^n$ let w_j denote the j^{th} coordinate of w and let $|w|$ denote the standard Euclidean norm. If Y is a compact set in $\mathbf{C}^n$, then the *polynomial (convex) hull* $\widehat{Y}$ of Y is given by

$$\widehat{Y} = \{z \in \mathbf{C}^n \big| |P(z)| \leq \sup_{w \in Y} |P(w)| \text{ for all polynomials } P \text{ on } \mathbf{C}^n\}.$$

A basic question concerning polynomial hulls is the following: given Y and $y \in \widehat{Y} \setminus Y$, does there exist an analytic variety through y with boundary in Y? If so, that variety gives a reason why y is in $\widehat{Y}$, by virtue of the maximum modulus principle for analytic varieties.

Let Δ denote the closed unit disk in $\mathbf{C}$ and Γ its boundary. Let B_n be the open unit ball in $\mathbf{C}^n$ and $\vec{0}$ the origin in $\mathbf{C}^n$. Let $K \subset \Gamma \times \mathbf{C}^n$. Does $\widehat{K}$ contain analytic varieties? Does $\widehat{K}$ contain graphs of analytic functions over the unit disk? Answers to these questions have been formulated, in works by several authors, in terms of the fibers $K_z \equiv \{w \in \mathbf{C}^n : (z,w) \in K\}$. In [AW,S2], it was shown that if the K_z are convex then $\widehat{K} \setminus K$ is the union of analytic graphs

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over the disk. The same was shown in the case $n = 1$ for fibers K_z which are connected and simply connected in [F1] and [S1]. (See also [HMa].) See [Če2] for a theorem in the case where the fibers are closures of completely circled pseudoconvex domains. Examples of K where $\widehat{K}$ does not contain analytic graphs over the disk have been constructed in [HMe] and [Če1]. In [HMa], Helton and Marshall conjecture that if the fibers K_z are (connected and) possess a property called *hypoconvexity* then the set $\widehat{K} \setminus K$ is the union of analytic graphs over the open disk.

We say that a subset of $\mathbf{C}^n$ is *hypoconvex* if its complement is the union of complex hyperplanes (of complex dimension $n-1$); *strictly hypoconvex* if it is smoothly bounded hypoconvex and the real Hessian of its defining function is positive definite on the complex tangent space at every point in the boundary. (The terminology in the literature on this subject varies; instead of hypoconvex, the reader will find "linearly convex," "lineally convex," "planarkonvex," and "complex geometric convexity" used to describe the same or similar notions.)

In [Wh] we proved Helton and Marshall's conjecture with some unnatural assumptions: that K can be smoothly deformed through sets $K^t \subset \Gamma \times \mathbf{C}^n$ such that for all t, K^t_z is strictly hypoconvex; that for t large, K^t_z is a ball of radius t and for t small, K^t_z shrinks around the origin in $\mathbf{C}^n$. Further, we assumed that the polynomial hull of K contains at least one analytic graph. Here we show that these assumptions are unnecessary by use of some constructions in work of Lempert [L1,L2,L3,L4,L5,L6]; we will show that the required deformation can be constructed. This is the subject of Theorem 1. Let $\Pi : \Delta \times \mathbf{C}^n \to \Delta$ be projection. We assume the following about K:

$$\text{(1)} \quad \left[\begin{array}{l} K \text{ is a } C^\infty \text{ submanifold of } \Gamma \times \mathbf{C}^n \text{ parametrized by a } C^\infty \text{ diffeomorphism } I : \\ \Gamma \times \partial B_n \to K,\ I(z,w) = (z, I_2(z,w)) \text{ for some } I_2, \text{ such that for every } z \in \Gamma,\ K_z \text{ is} \\ \text{the boundary of a bounded connected strictly hypoconvex open set } Y_z \text{ in } \mathbf{C}^n. \end{array} \right.$$

Theorem 1. *Suppose K satisfies (1). Then $\widehat{K} \cap \Pi^{-1}(\operatorname{int} \Delta)$ is either (a) empty, (b) the graph of a single analytic mapping $f : \operatorname{int} \Delta \to \mathbf{C}^n$ which extends to be in $C^\infty(\Delta)$ or (c) the union of infinitely many analytic graphs over the disk which extend to be in $C^\infty(\Delta)$. If $(z_0, w_0) \in \partial\widehat{K}$, $|z_0| < 1$, there is only one analytic $f \in H^\infty(\Delta)^n$ such that $f(z_0) = w_0$ and $f(z) \in \widehat{K}_z$ for a.e. $z \in \Gamma$. In case (c), we have that for all $z \in \Delta$, $\widehat{K}_z$ is hypoconvex with C^1 boundary.*

The smoothness statement in the last sentence can be improved; see [BL,Wh].

We use similar analysis to study another problem posed in [HMa]. Let $\rho : \Gamma \times \mathbf{C}^n \to \mathbf{R}$. A problem of H^∞ (frequency domain) design is to find $f \in H^\infty(\Delta)^n$ such that the essential supremum of $z \mapsto \rho(z, f(z))$ is as small as possible. (In H^∞ design, f determines a design for a machine and ρ measures something bad about that design that we wish to keep small. See [HMa] for a discussion of this.) The question we address here is whether a minimizer f is unique and whether it is smooth up to the boundary of Δ. Let

$$\gamma_\rho \equiv \inf_{f \in H^\infty(\Delta)^n} \operatorname*{ess\,sup}_{z \in \Gamma} \rho(z, f(z)). \tag{2}$$

Theorem 2. *Suppose that $\rho : \Gamma \times \mathbf{C}^n \to \mathbf{R}$ is continuous and that the set K where ρ equals γ_ρ satisfies (1), where $Y_z = \{w \in \mathbf{C}^n : \rho(z,w) < \gamma_\rho\}$. Then there exists a unique $\phi \in H^\infty(\Delta)^n$ such that $\rho(z,\phi(z)) \le \gamma_\rho$ for a.e. $z \in \Gamma$; in fact ϕ extends to be in $C^\infty(\Delta)$ and $\rho(z,\phi(z)) = \gamma_\rho$ for all $z \in \Gamma$. The set $\widehat{K} \cap \Pi^{-1}(\operatorname{int}\Delta)$ contains only the points on the graph of ϕ. If $\rho(\overline{z},\overline{w}) = \rho(z,w)$, then ϕ is $\mathbf{R}^n$-valued on the real axis, i.e. $\overline{\phi(\overline{z})} = \phi(z)$.*

This generalizes our own work [Wh, Theorem 3] and work of Vityaev [V, Theorem 1.5]. Vityaev assumes that an optimizer ϕ is smooth up to the boundary of Δ and proves that ϕ is unique in the class of analytic mappings $\operatorname{int}\Delta \to \mathbf{C}^n$ which extend to be smooth on the boundary.

We shall first prove a slightly weaker version of Theorem 1:

Theorem 3. *Suppose K satisfies (1) such that for every $z \in \Gamma$, Y_z contains the origin. Then $\widehat{K} \cap \Pi^{-1}(\operatorname{int}\Delta)$ is the union of graphs of analytic mappings $f : \operatorname{int}\Delta \to \mathbf{C}^n$ which extend to be in $C^\infty(\Delta)$. If $(z_0, w_0) \in \partial\widehat{K}$, $|z_0| < 1$, there is only one analytic $f \in H^\infty(\Delta)^n$ such that $f(z_0) = w_0$ and $f(z) \in \widehat{K}_z$ for a.e. $z \in \Gamma$. For all $z \in \Delta$, $\widehat{K}_z$ is hypoconvex with C^1 boundary.*

Remark. Theorem 3 is true if we replace the requirement that K enclose the zero graph by the requirement that it enclose some analytic graph, by holomorphic change of coordinates. Theorem 1 and Theorem 2 are known in the case $n = 1$, so we assume throughout that $n > 1$. We first prove Theorem 3, then Theorem 2, and then Theorem 1.

We discuss some notation. Let $Df(x)$ be the gradient of a function f at x and let $D^2f(x)$ be the real Hessian (i.e., the second derivative) at x. We write $D^2f(x)[u,v]$ to denote the value of the bilinear form $D^2f(x)$ at the pair u, v. Let ρ be a real valued function on an open $U \subset \Gamma \times \mathbf{C}^n$. Then $D_w\rho(z,w)$ is the vector $(\frac{\partial\rho}{\partial w_1}(z,w), \frac{\partial\rho}{\partial w_2}(z,w), ..., \frac{\partial\rho}{\partial w_n}(z,w))$. We say that $\rho : U \to \mathbf{R}$ is strictly hypoconvex on U if $D_w\rho \neq 0$ there and for every compact subset of U there exists $\kappa > 0$ such that whenever

$$\sum_{j=1}^{n} u_j \frac{\partial\rho}{\partial w_j}(z, w_1, w_2, ..., w_n) = 0$$

on that compact subset we have

$$D^2\rho(z, w_1, w_2, ..., w_n)[0, u_1, u_2, ..., u_n][0, u_1, u_2, ..., u_n] \ge \kappa|u|^2.$$

In Theorem 1 and Theorem 2, the fibers K_z of K enclose bounded connected strictly hypoconvex sets which are also polynomially convex; see [Wh]. Work of Lempert shows that the compact set bounded by K_z is homeomorphic to $\overline{B}_n$ via a mapping : $\overline{B}_n \to \widehat{K}_z$ which is C^∞ except at the origin. (See also Corollary 4.6.9 and Theorem 4.6.12 of [Hö].) Since the closed set bounded by K_z is polynomially convex for $z \in \Gamma$, when we compute the polynomial hull of K, its fiber over $z \in \Gamma$ will be $(\widehat{K})_z = \widehat{K_z}$ by Lemma 1 of [HMa]. For ease in notation, we shall refer to this set as $\widehat{K}_z$.

We say that K is a level set of ρ if there exists a constant c such that K is the set of all points in the domain of ρ where ρ takes the value c. We shall frequently need a defining function for a set satisfying (1). Assuming that $\mathcal{K}$ satisfies (1) and $\mathcal{Y}_z$ is the bounded open set bounded by $\mathcal{K}_z$, here are properties of ρ that we shall need:

(3)
- (a) ρ is defined and C^∞ in a neighborhood of $\mathcal{K}$ in $\Gamma \times \mathbf{C}^n$, such that $D_w\rho(z,w) \neq 0$, $\mathcal{K}$ is a level set of ρ, and ρ is strictly hypoconvex on its domain;
- (b) The values of $\rho(z,\cdot)$ are (strictly) smaller in $\mathcal{Y}_z$ than on $\mathcal{K}_z$.

For our original K, we may find such a ρ in the following manner: let $d_z(w)$ be the distance from w to K_z. Then for (z,w) near K, let $\rho(z,w)$ be $1 + d_z(w)$ for w in the unbounded component of $(K_z)^c$ and $1 - d_z(w)$ for w in the bounded component of $(K_z)^c$. This ρ is C^∞ and strictly hypoconvex for (z,w) sufficiently close to K; we can then extend ρ to $\Gamma \times \mathbf{C}^n$, so that ρ satisfies (3).

We seek a ρ which satisfies

(4)
- (a) $\rho : \Gamma \times \mathbf{C}^n \to [0,\infty)$ is continuous, and C^∞-smooth where $\rho \neq 0$;
- (b) there exists an $R > 1$ such that if $0 < \rho(z,w) \leq R$ then $D_w\rho(z,w) \neq 0$ and ρ is strictly hypoconvex as in (3) where ρ is smooth;
- (c) for every t, $0 < t \leq R$, the set K^t where $\rho = t$ is compact, with fibers K^t_z diffeomorphic to ∂B_n, where B_n is the open unit ball in $\mathbf{C}^n$;
- (d) $K = K^1$;
- (e) $\{(z,w) | \rho(z,w) = R\} = \{(z,w) | |w| = R\}$ and $\rho(z,w) > R$ if $|w| > R$;
- (f) There exists a continuous function $S(z)$, $|S(z)| < R$, such that $\{(z,w) \in \Gamma \times \mathbf{C}^n | w = S(z)\} = \{(z,w) \in \Gamma \times \mathbf{C}^n | \rho(z,w) = 0\}$.

(Such a ρ will satisfy (3) on p.679 of [Wh].)

§2 **Lempert's work.**

Let $A(\Delta) = \{f : \Delta \to \mathbf{C} \,|\, f \text{ is continuous on } \Delta \text{ and analytic in int}\,\Delta\}$. We shall rely on constructions contained in work of Lempert, which we shall summarize. (See [L1,L2,L3,L4,L5,L6].) This work will allow us to deform the fibers K_z of our set K in a manner which is necessary to obtain a defining function satisfying (4).

Fix any $z \in \Gamma$. As observed earlier, the interior of $\widehat{K}_z$ is a closed C^∞-bounded strictly hypoconvex domain Y_z. Let $\tau_z : \widehat{K}_z \to \mathbf{R}$ be Lempert's solution to the homogeneous complex Monge-Ampère equation for $\widehat{K}_z$ with logarithmic singularity at $S(z)$ ($\tau_z(w) = \log|w - S(z)| + O(1)$ as $w \to S(z)$) and boundary value 0 on K_z. (See [L2,L3].) Let $u_1(z,w) \equiv (u_1)_z(w) = e^{2\tau_z(w)}$. According to Lempert's work, given any pair (p,ν) with p in int $\widehat{K}_z$ and $\nu \in \mathbf{C}^n$, $|\nu| = 1$, there exists a unique extremal analytic disc, some of whose properties we shall need (see [L2]). An extremal disc satisfies the following (among other things): it is a continuous $f_{z,\nu} : \Delta \to \widehat{K}_z$ such that $f_{z,\nu}$ is analytic on int Δ , $f_{z,\nu}(0) = p$, $f'_{z,\nu}(0)$ is a positive multiple of ν, $f_{z,\nu}(\partial\Delta) \subset K_z$ and $f_{z,\nu}$ has a holomorphic left inverse $F_{z,\nu}$ which is C^∞ on $\widehat{K}_z$ (see Proposition 1 of [L2] and its proof). We will let $p = S(z)$. The level sets of $F_{z,\nu}$ are complex hyperplanes (intersected with $\widehat{K}_z$). Each of these complex hyperplanes meets the extremal disc in exactly one point([L2], Proposition 1 and its proof). At this point, the complex hyperplane is tangent to the level set of $(u_1)_z$ through that point. On the extremal disc parametrized by $f_{z,\nu}$, $\log|F_{z,\nu}(w)| = \tau_z(w)$, so $|F_{z,\nu}(w)|^2 = e^{2\tau_z(w)} = u_1(z, w)$. Furthermore, on $\widehat{K}_z$, $\tau_z(w) = \max_{\nu\in C^n,|\nu|=1} \log|F_{z,\nu}(w)|$. For $0 < c \leq 1$ the set $\{w \in \widehat{K}_z | \tau_z(w) \leq c\}$ is homeomorphic to the closed ball via a mapping which is C^∞ except at p; Lempert constructs this homeomorphism in [L3] for convex domains. It is also true (see the Reflection Principle in [L2, p.348]) that the extremal disks for K_z such that $f(0) = S(z)$ extend to be in $C^\infty(\Delta)$. For $0 < u_1 \leq 1$, u_1 is strictly hypoconvex. ([L4,pp.523,576])

We also make use of Lempert's solution $\tilde{\tau}_z$ to the homogeneous complex Monge-Ampère equation on the unbounded component of $(K_z)^c$, (with boundary value 0 on K_z). (See [L5, p.881] and the remark extending the result to hypoconvex domains on p.884.) Let $u_2(z, w) = e^{2\tilde{\tau}_z(w)}$. Then u_2 extends continuously to K, attaining the boundary value 1. ([L5,pp.881,884]) Where $u_2 > 1$, u_2 is strictly hypoconvex.([L5,p.883])

§3 **Deformation of the set** K.

Lemma 1 . *If K satisfies (1), there exists a C^∞ mapping $S : \Gamma \to \mathbf{C}^n$ such that $S(z) \in Y_z$ for all $z \in \Gamma$.*

Proof. This follows almost immediately from the definition of K. Fix $q \in \partial B_n$. There we have that $I_2(z, q) \in K_z$ for all $z \in \Gamma$; let $n(z, w)$ denote the inward pointing unit normal to K_z at $w \in K_z$. Since $I_2(z, q)$ is C^∞ in z, we may let $S(z) = I_2(z, q) + rn(z, I_2(z, q))$ if r is chosen sufficiently small. (This 'pushes' $I_2(z, q)$ a short distance into Y_z.) □

We assume $S(z)$ is any function satisfying the conditions of Lemma 1 . For $\epsilon > 0$, fixed $z \in \Gamma$, $\nu \in \mathbf{C}^n$, let $f_{z,\nu} : \Delta \to \widehat{K}_z$ be the extremal mapping with $f(0) = S(z)$ and $f'(0) = \lambda\nu$, $\lambda > 0$. Then let $F_{z,\nu}$ be the holomorphic left inverse to $f_{z,\nu}$ as defined in [L2], and let $F^\epsilon_{z,\nu}(w)^2 = |F_{z,\nu}(w)|^2 + \epsilon|w - f_{z,\nu}(F_{z,\nu}(w))|^2$.

Lemma 2 . *There exists $\epsilon > 0$ such that for all $z \in \Gamma$, on the set $\{w \in \widehat{K}_z : (u_1)_z(w) \leq 1/2\}$ we have that $(u_1)_z(w)$ is the maximum*

$$\max_{\nu\in C^n,|\nu|=1} (F^\epsilon_{z,\nu})(w)^2.$$

(The set $\{w \in \widehat{K}_z : (u_1)_z(w) \leq 1/2\} \subset\subset$ int $\widehat{K}_z$.)

Proof. The maximum above exists because the quantity there depends continuously on ν, which varies over a compact set. Let $\nu \in \mathbf{C}^n$, $|\nu| = 1$. If w is in the image of $f_{z,\nu}$, then $F^\epsilon_{z,\nu}(w)^2 = |F_{z,\nu}(w)|^2$ since $w = f_{z,\nu}(F_{z,\nu}(w))$. The function $(u_1)_z(w)$ is less than or equal to $\max_{\nu\in C^n,|\nu|=1}(F^\epsilon_{z,\nu})(w)^2$ because Lempert proves that $\tau_z(w) = \max_{\nu\in C^n,|\nu|=1} \log|F_{z,\nu}(w)|$, so $(u_1)_z(w) = \max_{\nu\in C^n,|\nu|=1} |F_{z,\nu}(w)|^2 \leq \max_{\nu\in C^n,|\nu|=1}(F^\epsilon_{z,\nu})(w)^2$. To prove the reverse inequality, we choose ϵ so small that 2ϵ is less than $D^2u_1(z, w)[v, v]$, where $z \in \Gamma$, $1/3 \leq (u_1)_z(w) \leq 2/3$, $|v| = 1$ and v is a complex tangent at w to the surface $\{w : (u_1)_z(w) = c\}$, for some c, $1/3 \leq c \leq 2/3$. (This is possible since u_1 is C^∞ and strictly hypoconvex where $0 < u_1 < 1$.) Then for each $f_{z,\nu}$ with holomorphic left inverse $F_{z,\nu}$, we claim $|F_{z,\nu}(w)|^2 + \epsilon|w - f_{z,\nu}(F_{z,\nu}(w))|^2 \leq (u_1)_z(w)$ in a neighborhood of $f(\{\lambda \in \mathbf{C} \,||\lambda| = 1/\sqrt{2}\}) \subset \{w \in \widehat{K}_z : (u_1)_z(w) = 1/2\}$. To see why the claim holds, first note that if w is in the image of $f_{z,\nu}$, then $w = f_{z,\nu}(\lambda)$ for some λ in the open unit disk; substituting $w = f_{z,\nu}(\lambda)$ into $\epsilon|w - f_{z,\nu}(F_{z,\nu}(w))|^2$ yields zero. Then from Lempert's work we already know that $|F_{z,\nu}(w)|^2 = (u_1)_z(w)$ for w in the image of f. Thus the claim holds - in fact, is an equality - for w in the image of $f_{z,\nu}$. Now let w be a point of the form $f_{z,\nu}(\lambda)$ for some λ such that $\sqrt{\frac{1}{3}} \leq |\lambda| \leq \sqrt{\frac{2}{3}}$. Then the second derivative of $u_1(z, \cdot) = \exp(2\tau_z(\cdot))$ at w in a direction v along a level set of $F_{z,\nu}$ is $> 2\epsilon$. The first derivatives of $u_1(z, \cdot)$ and $F_{z,\nu}(w)^2$ at w in the direction of v are both zero. (We recall that a level set of $F_{z,\nu}$ is an affine complex hyperplane tangent to a level set of $(u_1)_z$ at a point in the image of $f_{z,\nu}$. Every point in the image of $f_{z,\nu}$ lies on precisely one of these level sets of $F_{z,\nu}$.) By comparing derivatives of functions, elementary calculus allows us to conclude that $|F_{z,\nu}(x)|^2 + \epsilon|x - f_{z,\nu}(F_{z,\nu}(x))|^2 < (u_1)_z(x)$ if $\sqrt{\frac{1}{3}} \leq |F_{z,\nu}(x)| \leq \sqrt{\frac{2}{3}}$ and x is sufficiently close to the image of $f_{z,\nu}$. We may then choose a $\delta > 0$ independent of ν and z such that for all (z, ν), $F^\epsilon_{z,\nu}(w)^2 \leq (u_1)_z(w)$ on a δ-neighborhood of $f_{z,\nu}(\{\lambda \in \mathbf{C} \,||\lambda| = 1/\sqrt{2}\})$; the above argument achieves this in a neighborhood of a fixed (z, ν). Then the continuity of the functions involved and a compactness argument obtains the δ for all $(z, \nu) \in \Gamma \times \partial B_n$. We may further choose ϵ even smaller that

$$F^\epsilon_{z,\nu}(w)^2 \leq (u_1)_z(w) \tag{5}$$

for any w *outside* the δ - neighborhood determined above independent of ν and z such that $u_1(z, w) = 1/2$. (We may do this for any fixed (z, ν) because on the topological sphere where $(u_1)_z(w) = 1/2$, $|F_{z,\nu}(w)|^2 < (u_1)_z(w)$ outside a δ-neighborhood of $f_{z,\nu}(\{\lambda \in \mathbf{C} \,||\lambda| = 1/\sqrt{2}\})$, so the ϵ can be found for this fixed (z, ν). This ϵ will work for (z', ν') nearby, by continuity. Since (z, ν) is allowed to vary in the compact set $\Gamma \times \partial B_n$, the ϵ can be found so that (5) holds, as desired.) Combining the two neighborhoods, we find that on the set where $(u_1)_z(w) = 1/2$, $F^\epsilon_{z,\nu}(w)^2 \leq (u_1)_z(w)$. We claim that this inequality extends to the region where $u_1 \leq 1/2$. We have $|F_{z,\nu}(w)|^2 + \epsilon|w - f_{z,\nu}(F_{z,\nu}(w))|^2 \leq (u_1)_z(w)$ if $(u_1)_z(w) = 1/2$. By Lempert's work, if $w \in Y_z \equiv int\, \widehat{K}_z$, there is a $\mu \in \mathbf{C}^n$, $|\mu| = 1$ such that $w = f_{z,\mu}(\lambda)$, $|\lambda| < 1$, for some extremal $f_{z,\mu}$; $F_{z,\mu}$ is the holomorphic left inverse of $f_{z,\mu}$. Then $(u_1)_z(f_{z,\mu}(\lambda)) = |\lambda|^2$, since if $w = f_{z,\mu}(\lambda)$, $\tau_z(f_{z,\mu}(\lambda)) = \log|F_{z,\mu}(f_{z,\mu}(\lambda))| = \log|\lambda|$. Also, since $F_{z,\nu}(f_{z,\mu}(0)) = 0$ and $f_{z,\mu}(0) - f_{z,\nu}(F_{z,\nu}(f_{z,\mu}(0))) = \vec{0}$, we can write that for $|\lambda| = 1/\sqrt{2}$, $|\lambda|^2|A(\lambda)|^2 + |\lambda|^2|B(\lambda)|^2 \leq |\lambda|^2$, where A, B are analytic on the disk, $F_{z,\nu}(f_{z,\mu}(\lambda)) = \lambda A(\lambda)$, and $f_{z,\mu}(\lambda) - f_{z,\nu}(F_{z,\nu}(f_{z,\mu}(\lambda))) = \lambda B(\lambda)$. Then $|A(\lambda)|^2 + |B(\lambda)|^2 \leq 1$ for $|\lambda| = 1/\sqrt{2}$. Since the left side is subharmonic in λ, this inequality extends to the disk where $|\lambda| \leq 1/\sqrt{2}$. Multiplying back by the $|\lambda|^2$, we obtain the desired inequality on the image of $f_{z,\mu}$. Since every point w such that $u_1(z, w) \leq 1/2$ lies on such an extremal disk, we find that the claim holds: $F^\epsilon_{z,\nu}(w)^2 \leq (u_1)_z(w)$ if $u_1(z, w) \leq 1/2$. We let $(\tilde{u_1})_z(w) = \max_{\nu \in C^n, |\nu|=1} F^\epsilon_{z,\nu}(w)^2$. Thus what we have just done shows that $(\tilde{u_1})_z(w) \leq (u_1)_z(w)$ if $u_1(z, w) \leq 1/2$. As noted previously, $(u_1)_z(w) \leq (\tilde{u_1})_z(w)$ on $\widehat{K}_z$. Combining these, we have the desired result. □

Lemma 3. *There exists a $\delta > 0$ such that $u_1(z,w)$ is strictly convex in w for $0 < |w - S(z)| < \delta$ and all z; hence $\{(z,w) \in \Gamma \times \mathbf{C}^n | u_1(z,w) \leq \delta\}$ has strictly convex fibers over the circle.*

Proof. Let $f_{z,\nu}$ be the extremal disc for $\widehat{K}_z$ such that $f'(0)$ is a positive multiple of $\nu \in \mathbf{C}^n$ (see [L2, Theorem 3]) and $f(0) = S(z)$. Let $F_{z,\nu}$ be the holomorphic left inverse of $f_{z,\nu}$ and let $F^\epsilon_{z,\nu}(w)^2 = |F_{z,\nu}(w)|^2 + \epsilon|w - f_{z,\nu}(F_{z,\nu}(w))|^2$, where ϵ is as chosen in Lemma 2. We claim that there exists $\delta > 0$ independent of z, ν such that $(F^\epsilon_{z,\nu})^2$ is strictly convex on $B_n(S(z), \delta)$, the open ball of radius δ centered at $S(z)$. If not, choose $\delta_n \to 0$ and a corresponding sequence of $(F^\epsilon_{z_n,\nu_n})^2$ such that the claim is false. Passing to a limit, we find that there exists a pair (z,ν) such that $(F^\epsilon_{z,\nu}(w))^2$ is not strictly convex in w at $S(z)$. We show that this is impossible. We have that (a) the real Hessian of $|F_{z,\nu}|^2$ is nonnegative at the origin (since $|F_{z,\nu}|^2$ has a local minimum there) and positive in every direction except those complex directions orthogonal to ν (recall that $D_w F_{z,\nu}$ is nonzero, as the left inverse of $f_{z,\nu}$.) In the complex directions orthogonal to ν, F is identically zero. Further, (b) $\epsilon|w - f_{z,\nu}(F_{z,\nu}(w))|^2$ is strictly convex in those directions complex orthogonal to ν. Thus the sum $|F_{z,\nu}(w)|^2 + \epsilon|w - f_{z,\nu}(F_{z,\nu}(w))|^2$ is strictly convex at $S(z)$, a contradiction. Thus $u_1(z,\cdot)$ is strictly convex in w for sufficiently small $|w - S(z)| \neq 0$ as well, as a maximum of the functions $(F^\epsilon_{z,\nu})^2$ (Lemma 2). $\square$

In an earlier version of this paper, we noted that if a strictly hypoconvex domain were completely circled about the origin, then its Kobayashi balls about the origin would then be dilations of the domain. Then Lemma 3 shows that a strictly hypoconvex domain which is completely circled must be convex. We indicated then that there must be a more elementary proof of this fact. Miran Černe has provided precisely such a proof in [Če2].

We will now piece together various defining functions for K in order to obtain one which satisfies (4); we will need the following lemma regarding mollification of such functions.

Lemma 4. *Suppose that K satisfies (1), has defining functions ρ_1, ρ_2 which satisfy (3), $\rho_1 = \rho_2 = 1$ on K, and $|D_w\rho_1| > |D_w\rho_2|$ on K. Let $\rho(z,w) = \rho_1(z,w)$ if $\rho_1(z,w) > 1$, and $\rho(z,w) = \rho_2(z,w)$ if $\rho_1(z,w) \leq 1$. Let ϕ_ϵ be a C^∞ approximation to the convolution identity in $(z,w) \in \Gamma \times \mathbf{C}^n$ whose support has radius $< \epsilon$. Let $N(K,\epsilon) = \{(z,w) \in \Gamma \times \mathbf{C}^n : |\rho(z,w) - 1| \leq \epsilon\}$ and let $K^c_\epsilon \equiv \{(z,w) \in \Gamma \times \mathbf{C}^n : (\rho * \phi_\epsilon)(z,w) = c\}$. Then given any neighborhood M of K, there exists a neighborhood $N(K,\epsilon_1)$ of K contained in M and ϵ_2 sufficiently small such that $N(K,\epsilon_1/4) \subset \{(z,w) \in \Gamma \times \mathbf{C}^n : |\rho * \phi_\epsilon(z,w) - 1| < \epsilon_1/2\} \subset N(K,\epsilon_1)$ for $\epsilon \leq \epsilon_2$, and K^ϵ_c satisfies (1) for $c \in [1 - \epsilon_1/2, 1 + \epsilon_1/2]$. Furthermore, $\rho * \phi_\epsilon$ is strictly hypoconvex on the set $\{(z,w) \in \Gamma \times \mathbf{C}^n : |\rho * \phi_\epsilon(z,w) - 1| < \epsilon_1/2\}$.*

Note. An analogous result holds in the event that we only have $\rho_1 = \rho_2 = k$ on K for some real constant k; merely apply Lemma 4 to $\rho_1 - k + 1, \rho_2 - k + 1$, which both have value 1 on K.

Proof. Note that the gradient condition on ρ_1, ρ_2 guarantees that ρ is the maximum of ρ_1 and ρ_2 in some neighborhood $N(K, 2\epsilon_1) \subset M$ of K. We will convolve ρ with a close approximation $\phi_\epsilon(z,w)$ to the identity in $(z,w) \in \Gamma \times \mathbf{C}^n$, where $\epsilon > 0$. We assume ϵ_2 is small enough that ρ does not vary more than $\epsilon_1/4$ on a disk of radius ϵ_2 which meets $N \equiv N(K,\epsilon_1)$. Since $\rho * \phi_\epsilon$ converges locally uniformly to ρ as $\epsilon \to 0$, we obtain the statement that for ϵ_2 small, $N(K,\epsilon_1/4) \subset \{(z,w) \in \Gamma \times \mathbf{C}^n : |\rho * \phi_\epsilon(z,w) - 1| < \epsilon_1/2\} \subset N(K,\epsilon_1)$ for $\epsilon \leq \epsilon_2$. Now $D_w\rho_1$ and $D_w\rho_2$ are nonzero in N and have the same direction on K, so the unit normal to the level sets of ρ varies continuously near N. Assume ϵ_2 is small enough that on a ball of radius ϵ_2 which meets N, the angle of the unit normal to level sets of $\rho(z,\cdot)$ varies no more than $\pi/10$. Then $D_w(\rho * \phi_\epsilon)$ is nonzero near N for $\epsilon \leq \epsilon_2$. Let $T(z,w)$ be the complex tangent space in $\mathbf{C}^n$ to the level set of $\rho(z,\cdot)$ at w. Then T is continuous. By choosing ϵ even smaller, we may make the direction of $\frac{D_w(\rho*\phi_\epsilon)}{|D_w(\rho*\phi_\epsilon)|}$ as (uniformly) near to the directions $\frac{D_w\rho_1}{|D_w\rho_1|}$ (on $\{\rho_1 \geq 1\} \cap N$) and $\frac{D_w\rho_2}{|D_w\rho_2|}$ (on $\{\rho_1 \leq 1\} \cap N$) as desired. This implies that for ϵ small, level sets of $\rho * \phi_\epsilon$ which meet N have spherical fibers over Γ. For ϵ sufficiently small and $1 - \epsilon_1/2 \leq c \leq 1 + \epsilon_1/2$, K_ϵ^c may be parametrized by a diffeomorphism from $\Gamma \times \partial B_n$ by composing the map which parametrizes $\{(z,w) : \rho(z,w) = c\}$ with orthogonal projection along its normals to K_ϵ^c. Now $\rho_1(z,\cdot), \rho_2(z,\cdot)$ are strictly convex on $T(z,w)$ at w. In fact, for all $(z,w) \in N(K,\epsilon_1)$ there exists $\theta > 0$ such that they are strictly convex at w in any direction with an angle within θ of $T(z,w)$. Assume ϵ_2 so small that on a ball of radius ϵ_2 meeting N, the normal to $T(z,w)$ varies no more than $\theta/3$. Assume ϵ_2 so small that for $\epsilon \leq \epsilon_2$, the direction of $D_w(\rho * \phi_\epsilon)$ is within $\theta/3$ of $\frac{D_w\rho_1}{|D_w\rho_1|}$ on $\{\rho_1 \geq 1\} \cap N$ and $\frac{D_w\rho_2}{|D_w\rho_2|}$ on $\{\rho_1 \leq 1\} \cap N$. Since ρ_1, ρ_2 are C^∞ and strictly hypoconvex on N, this means that for ϵ_2 small enough, there exists $C > 0$ such that

$$\frac{\rho_i(z, w+hv) + \rho_i(z, w-hv)}{2} \geq \rho_i(z,w) + Ch^2, \tag{6}$$

for all $h \in \mathbf{R}$, $|h| \leq \epsilon_2$, (z,w) near N, $i = 1,2$ and v a unit vector in $\mathbf{C}^n$ whose direction is within θ of $T(z,w)$. The property (6) is preserved under taking the maximum of ρ_1, ρ_2, so ρ satisfies

$$\frac{\rho(z, w+hv) + \rho(z, w-hv)}{2} \geq \rho(z,w) + Ch^2, \tag{7}$$

for all $h \in \mathbf{R}$, $|h| \leq \epsilon_2$, (z,w) near N, $i = 1,2$ and v a unit vector in $\mathbf{C}^n$ whose direction is within θ of $T(z,w)$. For (z,w) near N, if v is within $\theta/3$ of being a tangent to the level set of $\rho * \phi_\epsilon(z,\cdot)$ at w, then it is within $2\theta/3$ of $T(z,w)$, so within θ of $T(z, w + h\tilde{v})$ if $|\tilde{v}| = 1$, $|h| \leq \epsilon_2$. Thus for (z_0, w_0) fixed near N,

$$\frac{\rho(z, w+hv) + \rho(z, w-hv)}{2} \geq \rho(z,w) + Ch^2, \tag{8}$$

for all $h \in \mathbf{R}$, $|h| \leq \epsilon_2$, (z,w) within ϵ_2 of (z_0,w_0), $i = 1,2$ and v a unit vector in $\mathbf{C}^n$ whose direction is within $\theta/3$ of the tangent space to the level set of $\rho * \phi_\epsilon(z_0,\cdot)$ at w_0. Then when convolving ρ with ϕ_ϵ, $\epsilon \leq \epsilon_2$, $\rho * \phi_\epsilon$ satisfies

$$\frac{\rho * \phi_\epsilon(z, w+hv) + \rho * \phi_\epsilon(z, w-hv)}{2} \geq \rho * \phi_\epsilon(z,w) + Ch^2, \tag{9}$$

for all $h \in \mathbf{R}$, $|h| \leq \epsilon_2$, (z,w) near N, $i = 1,2$ and v a unit vector in $\mathbf{C}^n$ at w whose direction is within $\theta/3$ of being tangent at w to the level set of $\rho * \phi_\epsilon(z,\cdot)$. This shows that $\rho * \phi_\epsilon$ is strictly hypoconvex in N. Since the level sets of $\rho * \phi_\epsilon$ were already shown to have spherical fibers over Γ, they now must enclose bounded strictly hypoconvex domains, so must satisfy (1), as desired. □

Lemma 5 . *Let K, ρ_1, ρ_2, ρ be as defined in Lemma 4 . Then given a neighborhood M of K there exists a strictly hypoconvex function ρ_3 with the same domain as ρ whose level sets satisfy (1) and which coincides with ρ outside of M.*

Note. Lemma 5 stills holds in the event that we only have $\rho_1 = \rho_2 = k$ on K for some real constant k; simply apply the lemma to the functions $\rho_1 - k + 1, \rho_2 - k + 1$.

Proof. Apply Lemma 4 to obtain $N = N(K, \epsilon_1)$ and ϵ_2. Using a partition of unity, we may write $\rho(z, w)$ as a sum of two functions Φ, Ψ, one of which (say Φ) has its support in $N(K, 4\epsilon_1/5) = \{(z, w) \in \Gamma \times \mathbf{C}^n : |\rho(z, w) - 1| \leq 4\epsilon_1/5\}$, and Φ is identically ρ in a neighborhood of $N(K, 3\epsilon_1/5)$. If ϵ_2 is small enough, we find $\Phi * \phi_\epsilon = \rho * \phi_\epsilon$ in $N(K, \epsilon_1/2)$, so is strictly hypoconvex there if $\epsilon \leq \epsilon_2$, as indicated in Lemma 4 . Furthermore, K^c_ϵ satisfies (1) for $c \in [1 - \epsilon_1/2, 1 + \epsilon_1/2]$. But if ϵ_2 is even smaller, $\Phi_\epsilon + \Psi$ is strictly hypoconvex outside of $N(K, \epsilon_1/4)$ for $\epsilon \leq \epsilon_2$ because ρ is strictly hypoconvex outside of $N(K, \epsilon_1/4)$, its level sets satisfy (1), and $\Phi_\epsilon \to \Phi$ in local C^2 norm. Then $\rho_3 \equiv \Phi_\epsilon + \Psi$ satisfies the conditions of the lemma for $\epsilon \leq \epsilon_2$. □

Let us recall the function $S(z)$ whose existence was asserted in Lemma 1 .

Lemma 6 . *Let K satisfy (1). Then there exists a strictly hypoconvex defining function u_2 defined in a neighborhood of $\{(z, w) \in \Gamma \times \mathbf{C}^n : w \in \widehat{K}_z\}$ which is strictly hypoconvex where it is nonzero, and such that for t sufficiently small, $\{w \in \mathbf{C}^n : u_2(z, w) = t\}$ is a sphere of radius independent of $z \in \Gamma$ centered at $S(z)$. The zero set of $u_2(z, \cdot)$ on $\widehat{K}_z$ consists only of $S(z)$, and $\{(z, w) \in \mathbf{C}^n : u_2(z, w) = t\}$ satisfies (1) for $t > 0$.*

Proof. Lempert [L2, Prop. 13] proves the existence of a C^∞ mapping $\Xi : \Gamma \times \mathbf{C}^n \to A(\Delta)^n$ such that $\Xi(z, \nu)$ is the extremal disc ϕ for $\widehat{K}_z$, where $\phi(0) = S(z)$ and $\phi'(0)$ is a positive multiple of ν. Composing with evaluation at the real number $r \in \Delta$ (denoted $\Xi(z, \nu)(r)$), $(z, \nu) \mapsto (z, \Xi(z, \nu)(r))$ becomes a C^∞ parametrization of the set $\{(z, w) \in \Gamma \times \mathbf{C}^n : u_1(z, w) = r^2\}$. (See §2 for the definition of u_1.) Let $\Xi^{(n)}$ denote the n^{th} derivative of Ξ. Now $\lambda \mapsto \Xi^{(n)}(z, \nu)(\lambda)$ is analytic on the open disk and extends smoothly to the boundary. The analyticity guarantees that

$$\left| \frac{\partial}{\partial r} \Xi^{(n)}(z, \nu_0)(r)|_{r=r_0} \right| = \left| \frac{\partial}{\partial (i\nu)} \Xi^{(n)}(z, \nu)(r_0)|_{\nu=\nu_0} \right|,$$

where i is the imaginary unit. For fixed n, the right side is uniformly bounded in r_0 for r_0 near 1, since Ξ is C^∞, so the left side is also. Thus $\Xi^{(n)}(z, \nu)(r) \to \Xi^{(n)}(z, \nu)(1)$ uniformly in (z, ν) as $r \to 1$. Now let ρ be the defining function for K prescribed after (3). Choose $R < 1$ so large that for $r \geq R$, the image $(z, \nu, r) \mapsto (z, \Xi(z, \nu)(r))$ lies in the domain where ρ is strictly hypoconvex and where projection to K_z is nonsingular. Let $\pi(\Xi(z, \nu)(r))$ be the point on K_z nearest $\Xi(z, \nu)(r)$; this is C^∞ in (z, ν, r) for r near 1. For $\epsilon > 0$ and $R \leq r < 1 + \epsilon$, define

$$\Xi_1(z, \nu)(r) = \frac{(1 - r)(\Xi(z, \nu)(R)) + (r - R)\pi(\Xi(z, \nu)(r))}{1 - R}.$$

Then if R is near enough to 1 and ϵ close enough to 0, the mapping $(z, \nu, r) \mapsto (z, \Xi_1(z, \nu)(r))$ is a C^∞ diffeomorphism in a neighborhood of the set where $R \leq r \leq 1$. For fixed r, the image of $(z,\nu) \mapsto (z, \Xi_1(z,\nu)(r))$ satisfies (1); this requires that ϵ be small enough, that R be close enough to 1, and makes use of the fact that all derivatives of $\Xi(\cdot,\cdot)(r)$ converge uniformly as $r \to 1$. Since $(z,\nu,r) \mapsto (z, \Xi_1(z,\nu)(r))$ is a diffeomorphism, we may define $u_3(z,w)$ to be the square of the r coordinate of the inverse. The functions u_1, u_3 coincide on the set where $u_1 = R^2$. Let $u_4 = f \circ u_3$, where f is an increasing C^∞ diffeomorphism of $[R^2, 1]$ to itself such that the derivative of f at R^2 is so large that the gradient $|D_w u_4|$ on the set where $u_1 = R^2$ is greater than $|D_w u_1|$. If we define u_5 to equal u_1 where $u_1 \leq R^2$ and u_4 where $u_1 \geq R^2$, then u_5 would satisfy all the properties we wish u_4 to satisfy except that it is not smooth on $\mathcal{S} := \{(z,w) \in \Gamma \times \mathbf{C}^n : u_5(z,w) = R^2\}$. We correct this problem by mollifying u_5 near $\mathcal{S}$. We do this by applying Lemma 5, with $\rho_1 = u_1 - R^2 + 1$, $\rho_2 = u_5 - R^2 + 1$, $K = \mathcal{S}$, on a small neighborhood N of $\mathcal{S}$. This produces the function u_6.

Next we modify u_6 to obtain a function whose level sets near the zero set have spherical fibers over the circle.

Choose P so small that the image $(z,\nu)(r) \mapsto (z, \Xi(z,\nu)(r))$ has strictly convex fibers over the circle for $r \leq P$ (possible by Lemma 3.) Choose $T < P$ so small that the absolute value of the w-coordinate of $\Xi(z,\nu)(P)$ is larger than T for all z, ν. Now define a function $u_7^*(z,w)$ as follows. The ray from $(z,0)$ through (z,w) pierces the sphere where $|w|^2 = T^2$ and the set where $u_6(z,\cdot)$ has the value P^2. Let us define u_7^* to be T^2 where the sphere is pierced, P^2 where the P^2 level set of u_6 is pierced, and extend affinely to the whole ray. Then u_7^* is strictly hypoconvex and its level sets satisfy (1) where $T^2 \leq u_7^* \leq P^2$. Let I be an increasing diffeomorphism of $[T^2, P^2]$ to itself such that $|D_w(I \circ u_7^*)(z,w)| \geq |w|$ when $|w| = T$ and $|D_w(u_6)(z,w)| \geq |D_w(I \circ u_7^*)(z,w)|$ when $u_6(z,w) = P^2$. Next define $u_7(z,w)$ to be $u_6(z,w)$, if $u_6(z,w) \geq P^2$; $|w|^2$, if $|w| \leq T$; and $I \circ u_7^*(z,w)$ for all other (z,w). Then u_7 is smooth except where $u_7 = T^2$ or P^2. We then locally mollify u_7 near these sets to obtain the desired defining function u_2 - using Lemma 5. Then u_2 is strictly hypoconvex and satisfies the conditions of the lemma. □

Lemma 7. *Let K satisfy (1). Then there exists a defining function ρ for K satisfying (4).*

We will also define $K^t \equiv \{(z,w) \in \Gamma \times \mathbf{C}^n : \rho(z,w) = t\}$.

Proof. With u_2 as in Lemma 6, define $L^t \equiv \{(z,w) \in \Gamma \times \mathbf{C}^n : u_2(z,w) = t\}$. We make use of a transformation provided by Lempert [L5, p.882, (5.3)]. Given a strictly hypoconvex, bounded, C^∞ bounded domain $D \subset \mathbf{C}^n$ there exists a "dual" $D' \subset \mathbf{C}^n$ which is also strictly hypoconvex, bounded, and C^∞ bounded. For ϵ small enough that $L^{1+\epsilon}$ is defined, we construct the dual of $\widehat{L}_z^{1+\epsilon}$ for every z and call its boundary H_z; the H_z fit together to form a new compact set H satisfying (1). We may construct a function $u_2{}'$ for H as u_2 was for K in Lemma 6. Consider the mapping

$$I : \Gamma \times \mathbf{C}^n \to \Gamma \times \mathbf{C}^n$$

$$(z,w) \mapsto \left(z, \frac{\frac{\partial u_2{}'}{\partial w_j}(z,w)}{\sum_{j=1}^n w_j \frac{\partial u_2{}'}{\partial w_j}(z,w)}\right).$$

Lempert's work shows that this is a C^∞ homeomorphism from $\{(z,w) \in \Gamma \times \mathbf{C}^n : w \in \widehat{H}_z \backslash \{0\}\}$ to $\{(z,w) \in \Gamma \times \mathbf{C}^n : w \notin int\, \widehat{L}_z^{1+\epsilon}\}$ (the "outside" of $\widehat{L}^{1+\epsilon}$.) We may then transform the function $u_2{}'$ to a function u_8 on $\{(z,w) \in \Gamma \times \mathbf{C}^n : w \notin int\, \widehat{L}_z^{1+\epsilon}\}$: $u_8(z,w) = 1/u_2{}'(I^{-1}(z,w))$.

Then the same work shows that the level sets of u_8 are compact sets satisfying (1), u_8 is identically 1 on $L^{1+\epsilon}$, $u_8 \geq 1$, u_8 is strictly hypoconvex when $u_8 > 1$, and $u_8 \to \infty$ as $|w| \to \infty$. Furthermore, when t is large enough, the set where $u_8 = t$ has spherical fibers, because the dual of a ball centered at the origin is another ball centered at the origin. Taking u_2 and $(1+\epsilon)u_8$ together produces a real valued function (call it u_9) defined on all of $\Gamma \times \mathbf{C}^n$ which is strictly hypoconvex except that it is not necessarily smooth when $u_9 = 1+\epsilon$. However, a mollification near $L^{1+\epsilon}$ - using Lemma 5 again - will produce a ρ satisfying (4), except possibly for (4)(e). But (4)(e) can easily be obtained by making use of the fact that the level sets of ρ have spherical fibers for large w, with radius depending only on z. □

§4 **Proofs of the theorems.**

Proof of Theorem 3. Lemma 7 shows that a K satisfying (1) has a defining function satisfying property (3) on page 679 of [Wh]. Theorem 3 then follows entirely from Theorem 2 of [Wh], except for the part about smooth extension to the boundary of Δ. This follows from the fact that K is a C^∞ manifold: we use results of Čirka [Či] and the remark in the last sentence in section 2 of [Wh, p. 694]. □

Proof of Theorem 2. From Lemma 7, we obtain a defining function for K which satisfies condition (3) on p.679 of [Wh]; hence we may apply Theorem 3 of [Wh] to conclude that ϕ exists and is unique. The function ϕ extends to be in $C^\infty(\Delta)$ by the Čirka result just mentioned. The only remaining fact that needs to be proven is the statement about the polynomial hull of K. To prove this, note that the hull of K^t increases with t; furthermore, $\widehat{K} = \cap_{t>1} \widehat{K}^t$. By Theorem 3, given any point (z_0, w_0) in $\widehat{K}$ such that $|z_0| < 1$, we have that (z_0, w_0) lies on the graph of some analytic $\mathbf{C}^n$ valued mapping f^t whose graph lies in $\widehat{K}^t$. Any local uniform limit f of a subsequence of the f^t must satisfy $f(z) \in \widehat{K}_z$ for a.e. $z \in \Gamma$ by [HMa,Corollaries 1,2], so $f = \phi$. This shows that the point (z_0, w_0) must lie on the graph of ϕ, so the graph of ϕ is $\widehat{K} \cap \Pi^{-1}(\text{int}\,\Delta)$. □

Proof of Theorem 1. Let ρ be as in Lemma 7, and let K^t be as before. Recall the definition of γ_ρ from (2). We consider 3 cases: (a)$\gamma_\rho < 1$,(b)$\gamma_\rho = 1$,(c)$\gamma_\rho > 1$. In case (a), by Theorem 2 there exists $f \in A(\Delta)^n$ such that $\rho(z, f(z)) < 1$ for $z \in \Gamma$. In fact f may have polynomial coordinates, by approximation. Thus K has the form of the K in Theorem 3 (after a holomorphic change of variable moving the graph of f to the graph of the zero function) so Theorem 1 will hold in case (a). In case (b), we have precisely the circumstances of Theorem 2, so Theorem 1 holds. In case (c), there exists $t > 1$ such that $\gamma_\rho = t$. From Theorem 2, we obtain a unique ϕ such that $\rho(z, \phi(z)) \leq \gamma_\rho$. The graph of this ϕ contains the only points which are in $\widehat{K}^t \cap \Pi^{-1}(\text{int}\,\Delta)$, so these are the only points which could possibly be in $\widehat{K} \cap \Pi^{-1}(\text{int}\,\Delta)$. We claim the latter set is empty; if not, it must contain a point of the graph of ϕ, so it must contain the whole graph by a theorem of Oka. But then the points $\phi(z)$ for $z \in \Gamma$ must be in $\widehat{K}_z$. They are not because for $z \in \Gamma$, $\phi(z) \in K^t_z$, and $K^t_z \cap \widehat{K}_z = \emptyset$ since $t > 1$. □

Theorem 1 shows that as in the case $n = 1$ and as is the case for K with convex fibers over the circle, the polynomial hull of a K in $\Gamma \times \mathbf{C}^n$ with strictly hypoconvex fibers is either empty, a single analytic graph or the union of infinitely many analytic graphs.

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We refer the reader to [Wh] for a more complete list of references.

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