GLOBAL UNIQUENESS TESTS AND PERFORMANCE BOUNDS FOR H^∞ OPTIMA

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Abstract.

Optimization of sup-norm type performance functions over the space of H^∞ functions is central to the subject of H^∞ design, that is, design where stability is the key constraint. Problems with large amounts of plant uncertainty are often highly nonconvex and therefore may have many solutions. In this article, even for highly nonconvex problems, we give a test one can perform, once a local optimum f^* has been computed, to see if it is a global optimum. The uniqueness phenomena we discovered uses H^∞ properties heavily and are considerably stronger than what occurs in other types of general optimization. Also, even when f^* may not be a global optimum we give a way to use it to bound the best performance possible.

Uniqueness results are valuable for assuring an engineer that a local optimum obtained in a computer run is in fact a true global optimum. This can save a practitioner a lot of time and anguish in that it replaces the usual process of initializing an optimization run many times to see if it always goes to the same local optimum; and even after vast numbers of experiments never being sure.

One of the least intuitive properties of SISO (single input, single output) control is that a (local) optimum for a carefully set up H^∞ problem (cf. Theorem 9.4.1 in [10], [7]) even with large amounts of plant uncertainty is unique. Such problems are quite nonconvex so the fact is surprising. While the result is false in general for MIMO (multiple input, multiple output) control (cf. [8]), in this note we are describing MIMO situations where uniqueness holds.

The setting in this paper is simultaneous (Pareto) optimization of several competing performances $\Gamma_1, \ldots, \Gamma_\ell$ and we obtain uniqueness results[1] for its solutions.

Key words.

H^∞ control, Frequency response methods, Uniqueness, Pareto, Optimization, Multiple performances, IQC, QFT.

AMS subject classifications. 32, 49K

This paper analyzes a problem in which one optimizes performance functions over the space H^∞_N of bounded analytic vector-valued functions $f = (f_1, \ldots, f_N)$ defined on the unit circle, $\mathbf{T}$, where each coordinate function f_j belongs to $L^\infty(\mathbf{T})$ and extends to be analytic on the entire unit disk. Let $C(\mathbf{T})$ be the space of continuous complex-valued functions on the circle and let $C^1(\mathbf{T})$ be those elements in $C(\mathbf{T})$ with continuous first derivatives.

1.1. Definition of Pareto Optimum. The performance criteria we optimize are described in terms of nonnegative continuous functions Γ defined on $\mathbf{T} \times \mathbf{C}^N$. We are given positive functions $\Gamma_j(e^{i\theta}, z)$, $j = 1, \ldots, \ell$ for $\ell \leq N$ with $e^{i\theta} \in \mathbf{T}$ and $z \in \mathbf{C}^N$. For function $f \in H^\infty_N$ we define the ℓ performances

$$\gamma_j(f) := \sup_{e^{i\theta} \in \mathbf{T}} \Gamma_j(e^{i\theta}, f(e^{i\theta})), \quad j = 1, \ldots, \ell.$$

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[1] An announcement of some of these results appeared as *Global uniqueness tests for H^∞ optima,* in Proceedings of the 39th IEEE Conference on Decision and Control, Vol. 2, IEEE Press, Piscataway, NJ, pp. 1043–1048.

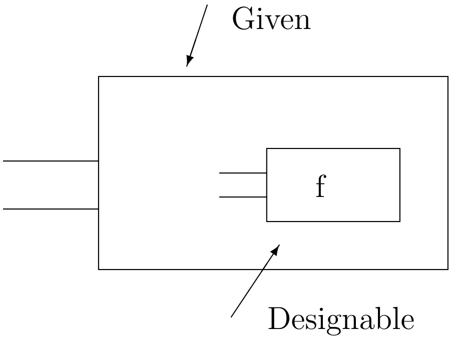

FIG. 1.1. *For a given plant we want to find the best designable part, represented by f*

The goals of this paper are best illustrated by the case of two performance functions Γ_1, Γ_2, even though all results hold for ℓ performance functions.

Definition *A function* $f^* \in H^\infty_N$ *is called a* Pareto optimum *for* Γ_1, Γ_2 *if for each* $f \in H^\infty_N$ *such that* $\gamma_1(f) \le \gamma_1(f^*)$ *and* $\gamma_2(f) \le \gamma_2(f^*)$, *we must have*

$$\gamma_1(f) = \gamma_1(f^*) \quad \text{and} \quad \gamma_2(f) = \gamma_2(f^*).$$

By the *MultiOPT* problem we shall mean the problem of finding a Pareto optimum. The definition for more Γ_j's is the obvious analogue.

The book of Boyd and Barrat [2] gives a good discussion of Pareto optimality. The paper [14] treats successfully a particular type of frequency domain Pareto optimality, superoptimal H^∞ optimization.

1.2. Engineering Motivation. This type of problem is central to frequency domain system design problems where stability is a key constraint. In particular it is important to the area of H^∞-control. The basic physical idea is simple. The following often occurs in a design procedure. We are required to build a system S but part of the system is given (we are stuck with it) and part of the system is designable (denote its frequency response function by f). Such a system is illustrated in Figure 1. The performance of the system S at frequency ω is a function $\Gamma(\omega, f(i\omega))$ which depends on ω and on our choice of the designable subsystem f. Let us take the convention that large Γ is bad while small Γ is good. Then in a worst case "broadband" design we consider the worst performance over all frequencies

$$\sup_\omega \Gamma(\omega, f(i\omega))$$

and try to minimize it over all admissible f. If our main constraint is that the designable subsystem f must be stable, then the design problem becomes the MultiOPT problem with only one Γ, after transforming the right half plane to the unit disk. In this paper we deal with the case where f consists of N designable subsystems $f_1, \cdots, f_N$, and where there are ℓ competing performance criteria $\Gamma_1, \Gamma_2, ..., \Gamma_\ell$. Nonconvex performance measures occur in problems with considerable plant uncertainty.

A number of authors, Mayne-Polak-et al [12], Fan-Tits-et al [4], Streit [17], Boyd-Barratt [2], Daleh, Pearson, Balas-Doyle-Glover-Packard-Smith [3], Helton-Merino

[10] and Sideris [16], have theory and computer programs on searching for an optimal f^* with certain kinds of Γ. The main H^∞ optimization problem of quantitative feedback theory (QFT) is essentially the MultiOPT problem. Also integral quadratic constraints (IQC's; see [13]) address such problems but in a different set of coordinates (behavioral coordinates). Multiple constraints in the frequency domain of a somewhat different flavor are in [11], [15] and [5]. There are similar physical problems that they can treat.

1.3. Geometric version of the problem. The MultiOPT problem can be stated geometrically in a way which is physically appealing. The sublevel sets

$$\mathcal{S}^j(\gamma_j) := \{(e^{i\theta}, z) \in \mathbf{T} \times \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma_j\}$$

$$\mathcal{S}^j_\theta(\gamma_j) := \{z \in \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma_j\}$$

of the performance functions Γ_j correspond to values of the frequency response function where the j^{th} performance measure is better (less) than γ_j. For fixed $\overrightarrow{\gamma} := (\gamma_1, \cdots, \gamma_\ell)$,

$$\mathcal{S}_\theta(\overrightarrow{\gamma}) := \mathcal{S}^1_\theta(\gamma_1) \cap \ldots \cap \mathcal{S}^\ell_\theta(\gamma_\ell) \qquad \forall e^{i\theta} \in \mathbf{T} \tag{1.1}$$

is the set of values simultaneously yielding performance level $(\gamma_1, \cdots, \gamma_\ell)$.

Given target sets $\mathcal{S}_\theta(\overrightarrow{\gamma})$ in $\mathbf{C}^N$, the suboptimal MultiOPT problem is to find a stable system f whose values $f(e^{i\theta})$ lie in the target sets

$$f(e^{i\theta}) \in \mathcal{S}_\theta(\overrightarrow{\gamma}).$$

Standard assumption. Assume that each Γ_j is three times differentiable. Assume that sets $\mathcal{S}_\theta(\overrightarrow{\gamma})$ have nonempty interiors for each $\overrightarrow{\gamma}$ and are uniformly bounded. Lastly assume that the sets are *uniformly contractible*: there exist mappings $I_t(e^{i\theta}, z)$ from $\mathcal{S}(\overrightarrow{\gamma})$ to $\mathcal{S}(\overrightarrow{\gamma})$ continuous in t, θ, z such that for each θ, $(e^{i\theta}, z) \mapsto I_t(e^{i\theta}, z)$ is the identity for $t = 0$, the first coordinate of $I_t(e^{i\theta}, z)$ is $e^{i\theta}$, and for each θ, $z \mapsto I_t(e^{i\theta}, z)$ is constant when $t = 1$. Intuitively, uniform contractibility just ensures that each of the domains $\mathcal{S}(\overrightarrow{\gamma})$ is arcwise connected (i.e., none have isolated components) and none of them contain holes. Clearly the class of $\mathcal{S}(\overrightarrow{\gamma})$ which are uniformly contractible contains the class of $\mathcal{S}(\overrightarrow{\gamma})$ whose $\mathcal{S}_\theta(\overrightarrow{\gamma})$ are all convex, a class of $\mathcal{S}(\overrightarrow{\gamma})$ upon which most uniqueness theory is based. Our assumption is weak and without it most theory and existing algorithms of any existing type appear impossible (unless one is in a situation where the holes do not matter and only one component matters.) [2]

1.4. The Gist of the Main Results. It is common for computer optimization algorithms at the k^{th} step to keep track of both the "primal variables" (in our case f^k) and "dual variables." These are called primal-dual algorithms. We shall see in Sections 2 and 4.2 that such an algorithm for MultiOPT which stops in a local

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[2] Roughly this means that for all θ, the set $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ consists only of one piece and has no holes. For such a set to contain holes or to be disconnected, one must be working with a highly nonlinear situation. Under these circumstances, even convergence of your computer runs to functions f^*, F can be problematic; possibly it would be worth while to reconsider the setup of the original problem.

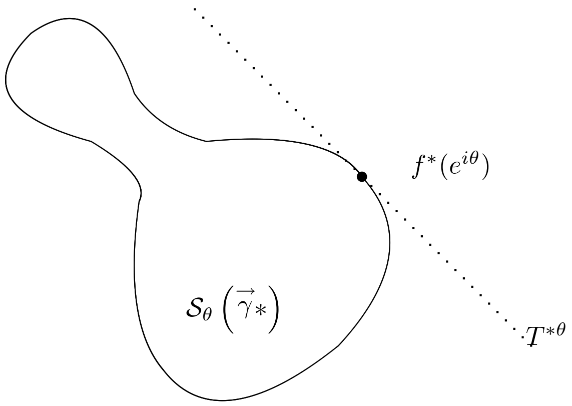

FIG. 1.2. *Non-convex but f^* is a unique global optimum.*

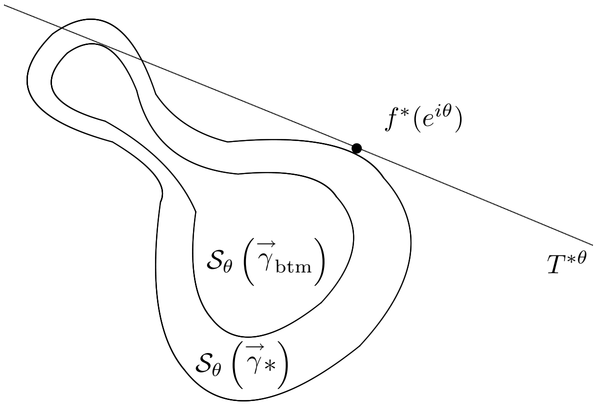

FIG. 1.3. *Non-convex but no performance is better than $\overrightarrow{\gamma}^{\text{btm}}$.*

optimum f^* with performance levels $\overrightarrow{\gamma}^{*} := (\gamma_1(f^*), \cdots, \gamma_\ell(f^*))$ produces information with the simple geometric interpretation that we know the primal local optimum f^* and a [3] tangent plane $T^{*\theta}$ to the boundary $\partial\mathcal{S}_\theta(\overrightarrow{\gamma}^{*})$ at $f^*(e^{i\theta})$. The tangent plane is a good way to visualize optimal "dual information."

Figure 2 illustrates this situation as well as our uniqueness test when the set $\mathcal{S}_\theta(\overrightarrow{\gamma}^{*})$ has smooth boundary. Figure 3 illustrates our lower bound for performance. These tests constitute our main results and roughly they say

> **For each θ, if the tangent plane $T^{*\theta}$ to $\mathcal{S}_\theta(\overrightarrow{\gamma}^{*})$ at $f^*(e^{i\theta})$**
>
> **(a) intersects $\mathcal{S}_\theta(\overrightarrow{\gamma}^{*})$ at only the point $f^*(e^{i\theta})$, then f^* is the unique global optimum for MultiOPT. See Figure 1.2.**
>
> **(b)does not intersect $\mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$, then any $f \in H^\infty_N$ has $\overrightarrow{\gamma}_j(f) \geq \overrightarrow{\gamma}^{\text{btm}}_j$ for some $j = 1, 2, \ldots, \ell$. That is, performance can be no better than $\overrightarrow{\gamma}^{\text{btm}}$. See Figure 1.3**.

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[3] When the set $\mathcal{S}_\theta(\overrightarrow{\gamma}^{*})$ has smooth boundary (as in problems with a single performance measure Γ^1) the tangent plane $T^{*\theta}$ is unique.

We emphasize that $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ need not be convex; our conditions are much less stringent. For comparison recall that a closed strictly convex set C with smooth boundary ∂C has the defining property that at each point z_0 on ∂C the tangent plane T_{z_0} to ∂C at z_0 intersects C only at z_0. Thus to check strict convexity one must check this property at *every point* z_0 on ∂C. Our test for uniqueness requires checking this condition at *only one point*, for each θ. This is a surprising property of optimization over spaces of analytic functions.

The tests work in more generality than described here. The sets $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ can have corners as would be the case with multi-performance problems; see Theorem 4.1. Also, a smaller set than the tangent plane, the *complex tangent plane* suffices in our test; see Theorem 4.2. Section 4.2 gives geometric interpretations of these theorems.

2. Optimality Conditions and Computation.

We begin the detailed description of our global uniqueness test and performance bound by saying precisely what is meant by primal and dual variables.

2.1. Primal -Dual Optimality Conditions.

Recall the optimality conditions for MultiOPT. First we introduce the notation

$$\frac{\partial \Gamma}{\partial z} = \begin{pmatrix} \frac{\partial}{\partial z_1}\Gamma_1 & \dots & \frac{\partial}{\partial z_N}\Gamma_1 \\ \dots & \dots & \dots \\ \frac{\partial}{\partial z_1}\Gamma_\ell & \dots & \frac{\partial}{\partial z_N}\Gamma_\ell \end{pmatrix} \tag{2.1}$$

Theorem 2.1. *([10],Theorem 17.1.1; [9])*

Assume Γ_j *for* $j = 1, \cdots, \ell$, *satisfies the standard assumption. Suppose that a continuous local optimum* f^* *does exist with performance values denoted*

$$\gamma_1^*, \cdots, \gamma_\ell^*$$

which makes

$$\frac{\partial \Gamma}{\partial z}(e^{i\theta}, f^*(e^{i\theta}))\frac{\partial \Gamma^T}{\partial z}(e^{i\theta}, f^*(e^{i\theta}))$$

invertible on all $e^{i\theta}$. *(This implies* $\ell \leq N$.*) Then there exist functions* ψ_j *in* $L^1(\mathbf{T})$, *for* $j = 1, \cdots, \ell$, *which satisfy*

Flatness: $\psi_j(e^{i\theta})(\gamma_j^* - \Gamma_j(e^{i\theta}, f^*(e^{i\theta}))) = 0$ *for* $j = 1, \cdots, \ell$.

Gradient Alignment:

$$\psi_1(e^{i\theta})\frac{\partial \Gamma_1}{\partial \bar{z}}(e^{i\theta}, f^*(e^{i\theta})) + \ldots + \psi_\ell(e^{i\theta})\frac{\partial \Gamma_\ell}{\partial \bar{z}}(e^{i\theta}, f^*(e^{i\theta})) = e^{-i\theta}\bar{F}(e^{i\theta}), \quad F \in H^2_N. \tag{2.2}$$

Normalization:

$$\sum_{j=1}^{\ell} \int_0^{2\pi} \psi_j d\theta = 2\pi$$

.

Positivity [4] *:* $\gamma_j^* - \Gamma_j(e^{i\theta}, f^*(e^{i\theta})) \geq 0$ *and* $\psi_j \geq 0$ *for* $j = 1, \cdots, \ell$.

Moreover, it is shown in [9] that if f^* *is once differentiable, the* ψ_j *are once differentiable.* Here A^T denotes the conjugate transpose of A and H_N^2 is the set of vector valued H^2 functions on the circle.

2.2. Computer optimization. It is common for computer optimization algorithms at the k^{th} step to keep track of both the "primal variables" (in our case f^k) and "dual variables" ψ_j^k and consequently F^k. We start with bad guesses $f^0 \in H_N^\infty, \psi_j^0$ and update them in various ways ultimately to approach a solution to the flatness and gradient alignment equations. At optimum a key property is that F is analytic. These are called primal-dual algorithms and are popular. (See [1], [18].) As we shall see in our H^∞ case the dual variable F^k has the interpretation that $e^{-i\theta}\bar{F}^k(e^{i\theta})$ is pointed "normally" to the sets $\mathcal{S}(e^{i\theta})$ at the point $f^k(e^{i\theta})$. Exactly what this means geometrically requires discussion (see Section 4.2) but it motivates calling the optimal dual vector function $e^{-i\theta}\bar{F}$ the *conjugate analytic normal* at f^*. The point is that many H^∞ optimization algorithms produce both a primal optimum f^* and a conjugate analytic normal $e^{-i\theta}\bar{F}(e^{i\theta})$ to $\mathcal{S}_\theta(e^{i\theta})$ at $f^*(e^{i\theta})$.

In summary, typical algorithms, c.f. [10], produce a sequence f_k, F_k of approximates to f, F where $f_k \in H_N^\infty \cap C^\infty(\mathbf{T})$. In some algorithms F_k does not belong to H_N^∞, but one can easily compute $\overset{\vee}{F}_k \in H_N^\infty$, which is the best L_N^2 approximate to F_k, and use it as a vague indicator of closeness to optimum f^*, F. In the next section we give a test which removes most of the vagueness from such diagnostics, in that for each $f_k, \overset{\vee}{F}_k$ it gives an absolute lower bound on the best possible performance $\overrightarrow{\gamma}^*$. The methods behind proving this lower bound lead to our global uniqueness result.

3. An Algorithmic Phrasing of our Uniqueness Test. We now describe our main result in a (high level) algorithmic format. Although a bit redundant it gives a casual reader a description of the method which (except for the most technical hypotheses) is self-contained. The subsequent sections give theorems supplying technical hypotheses and verify that the algorithm works.

Suppose you have run your favorite numerical algorithm for solving the optimization problem MultiOPT in Section 1.1 and that you have obtained a local optimum $f^*, \overrightarrow{\gamma}^*$ and the corresponding dual function F.

Before we describe our test, we need a few definitions. For any integer $N > 0$, the N dimensional complex vector space $\mathbf{C}^N$ has the usual inner product $\langle z, w\rangle_C := \sum_j z_j \bar{w}_j =: z \cdot \bar{w}$, but $\mathbf{C}^N$ can be viewed as a $2N$ dimensional real vector space with the inner product $\langle z, w\rangle_R := \mathrm{Re} \sum_j z_j \bar{w}_j =: \mathrm{Re}\,[z \cdot \bar{w}]$. Define the complex plane which is *complex orthogonal* to the vector $\mathbf{N}$ at location b by

$$\mathbf{N}^{c\perp}(b) := \{z \in \mathbf{C}^N : \bar{\mathbf{N}} \cdot (z - b) = 0\}. \tag{3.1}$$

This complex orthogonal is a subset of the ordinary *real orthogonal* complement

$$\mathbf{N}^{r\perp}(b) := \{z \in \mathbf{C}^N : \mathrm{Re}\,[\bar{\mathbf{N}} \cdot (z - b)] = 0\} \tag{3.2}$$

to $\mathbf{N}$ at b. The b will occasionally be omitted when it is clear from context which point b is intended.

Now we describe our test for global optimality of f^* and, if optimality does not hold, we describe a bound on the best performance.

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[4] This just reiterates the definition of γ_j^*

3.1. The Algorithm. Suppose we are given continuous functions f, F in H^∞_N, where F is never zero on the circle, and the Γ_j satisfy the standard assumption.

1. For each θ, compute [5] a linearly independent set of $N-1$ vectors $\phi^1(e^{i\theta}), \phi^2(e^{i\theta}), ..., \phi^{N-1}(e^{i\theta})$ satisfying the equation $\phi^k(e^{i\theta}) \cdot F(e^{i\theta}) = 0$ for $k = 1$ to $N-1$. (These $N-1$ vectors form a basis for $\mathbf{N}^{c\perp}_\theta(f(e^{i\theta}))$, where $\mathbf{N}_\theta = e^{-i\theta}\overline{F(e^{i\theta})}$.)

2. For each θ, define $v^j_\theta(w)$ by

$$v^j_\theta(w) := \Gamma_j(e^{i\theta}, f(e^{i\theta}) + w_1\phi^1(e^{i\theta}) + \ldots + w_{N-1}\phi^{N-1}(e^{i\theta})),$$

where $w = (w_1, w_2, ..., w_{N-1}) \in \mathbf{C}^{N-1}$.

3. For each θ and $j = 1, \ldots, \ell$, compute the minimum $\overset{\rightarrow\text{btm}}{\gamma}_j$ of $v^j_\theta(w)$ for $w \in \mathbf{C}^{N-1}$.

(a) **Global uniqueness of optimum.** If f^* is a local Pareto optimum for MultiOPT with performance $\overset{\rightarrow *}{\gamma}$, and F is the corresponding analytic dual, then if for each fixed θ the point $w_0 = 0$ in $\mathbf{C}^{N-1}$ is the unique nondegenerate [6] minimizer for

$$\tilde{\Gamma}(e^{i\theta}, w) := \max\left\{\frac{v^1_\theta(w)}{\gamma^*_1}, \frac{v^2_\theta(w)}{\gamma^*_2}, \ldots, \frac{v^\ell_\theta(w)}{\gamma^*_\ell}\right\}$$

then f^* is the only solution to MultiOPT achieving performance level $\overset{\rightarrow *}{\gamma}$.

(b) **The multi-performance $\overset{\rightarrow\text{btm}}{\gamma}$ is a bound on the best performance.** If there exists a $\mathbf{C}^N$-valued continuous[7] function q defined on the circle such that $\operatorname{Re}(e^{i\theta}F(e^{i\theta}) \cdot (q(e^{i\theta}) - f(e^{i\theta}))) < 0$ for all θ and $\Gamma_j(e^{i\theta}, q(e^{i\theta})) \leq \overset{\rightarrow\text{btm}}{\gamma}_j$ for all j and θ, [8] then for any $\tilde{f} \in C(\mathbf{T}) \cap H^\infty_N$, we have $\gamma_j(\tilde{f}) \geq \overset{\rightarrow\text{btm}}{\gamma}_j$ for at least one $j = 1, \ldots \ell$.

3.2. A Tutorial Example. Take

$$\Gamma(e^{i\theta}, z_1, z_2) = |(z_1 - e^{-i\theta})(z_1 - ke^{-i\theta})|^2 + |z_2|^2.$$

where $k = e^{\frac{3\pi i}{4}}$. It is easy to see that this is not a convex problem.

One finds that a local optimizer is $f^* := (0,0)$, giving optimal value equal to $|k|^2 = 1$. We wish to illustrate our global optimality test. Calculate that the partial of Γ with respect to z is

$$\frac{\partial\Gamma}{\partial z}(e^{i\theta}, z) = ((2z_1 - (k+1)e^{-i\theta})\overline{(z_1 - e^{-i\theta})(z_1 - ke^{-i\theta})}, \bar{z_2})$$

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[5] Given $F(e^{i\theta}) = (F_1(e^{i\theta}), F_2(e^{i\theta}), ..., F_N(e^{i\theta})) \in \mathbf{C}^N$, there are many ways to compute $\phi^k(e^{i\theta})$ for each θ. One way is to select $\phi^1(e^{i\theta}) = (-F_2(e^{i\theta}), F_1(e^{i\theta}), 0, 0, 0, ..., 0)$, $\phi^2(e^{i\theta}) = (-F_3(e^{i\theta}), 0, F_1(e^{i\theta}), 0, 0, 0, ..., 0)$, etc., provided that for that θ, we have $F_1(e^{i\theta}) \neq 0$. The Gram-Schmidt process may be used to obtain an orthogonal set of $\{\phi^k(e^{i\theta})\}_{k=1}^{N-1}$; this may help with numerics.

[6] Nondegeneracy is a very technical condition which will be defined later in Section 4; it involves so fine a distinction that we do not think one would check it in practice.

[7] Note that q does not have to be - in fact, must not be - analytic.

[8] Such a q would exist, for example, if for every j, $\operatorname{Re}(e^{i\theta}F(e^{i\theta}) \cdot (w - f(e^{i\theta}))) < 0$ for all θ and $w \in \mathcal{S}_\theta(\overset{\rightarrow\text{btm}}{\gamma})$. Here, uniform contractibility of $\mathcal{S}_\theta(\overset{\rightarrow\text{btm}}{\gamma})$ would guarantee that the continuous function to which those sets may be uniformly contracted would satisfy the conditions that q must satisfy.

which at $z = (0,0)$ is

$$\frac{\partial \Gamma}{\partial z}(e^{i\theta}, (0,0)) = -((k+1)\bar{k}e^{i\theta}, 0).$$

Note this function extends analytically to the disk, which illustrates the gradient alignment condition of local optimality Theorem 2.1; here $\psi_1 = 1$. There are different ways to implement our global optimality test and we illustrate several of them.

The subspace of $\mathbf{C}^N$ which is complex orthogonal to the conjugate of $\frac{\partial \Gamma}{\partial z}(e^{i\theta}, (0,0))$ is called the complex tangent plane, and at fixed $e^{i\theta}$ it is

$$\mathbf{N}_\theta^{c\perp}((0,0)) := \{z : z_1 = 0\}.$$

The key issue is whether it intersects

$$\mathcal{S}_\theta(1) = \{z : |(z_1 - e^{-i\theta})(z_1 - ke^{-i\theta})|^2 + |z_2|^2 \leq 1\}$$

in more than the one point $(0,0)$. The points in $\mathbf{N}_\theta^{c\perp}((0,0)) \cap \mathcal{S}_\theta(1)$ are

$$\{z : z_1 = 0 \text{ and } |(z_1 - e^{-i\theta})(z_1 - ke^{-i\theta})|^2 + |z_2|^2 \leq 1\} = \{z : z_1 = 0 \text{ and } |k|^2 + |z_2|^2 \leq 1\}.$$

Since $|k| = 1$ this forces $z_2 = 0$, so $z = (0,0)$. Therefore $(0,0)$ is the unique global optimizer for this MultiOPT problem.

We now give an argument analogous to that just given, but in terms of our algorithm above. A basis for $\mathbf{N}_\theta^{c\perp}((0,0))$ is $\phi^1(e^{i\theta}) = (0,1)$ for all θ. Then

$$v_\theta^1(w) = \Gamma(e^{i\theta}, (0,0) + w(0,1)) = |w|^2.$$

Its minimizer over $w \in \mathbf{C}$ is clearly $(0,0)$ and is unique and nondegenerate.

3.3. Theoretical Justification. The lower bound stated in the Algorithm can be rigorously proved under modest assumptions as we now see in Theorem 3.1. See Theorem 3.2 for another bound not requiring the hypothesis concerning q.

THEOREM 3.1. *Suppose the performance functions $\Gamma_1, \cdots, \Gamma_\ell$ satisfy the Standard Assumption. Suppose that $f, F \in H_N^\infty$ are continuous with F never vanishing. Set $\mathbf{N}_\theta := e^{-i\theta}\bar{F}(e^{i\theta})$. Suppose there exist $\overset{\rightarrow \mathrm{btm}}{\gamma_j}$ such that for every θ and $z \in \mathbf{N}_\theta^{c\perp}(f(e^{i\theta}))$, there exists a $j = 1, 2, ..., \ell$ such that*

$$\Gamma_j(e^{i\theta}, z) > \overset{\rightarrow \mathrm{btm}}{\gamma_j}. \tag{3.3}$$

Suppose that there exists some $\mathbf{C}^N$ valued continuous function $q(e^{i\theta})$ such that $\mathrm{Re}\,(e^{i\theta}F(e^{i\theta}) \cdot (q(e^{i\theta}) - f(e^{i\theta}))) < 0$ for all θ and $\Gamma_j(e^{i\theta}, q(e^{i\theta})) \leq \overset{\rightarrow \mathrm{btm}}{\gamma_j}$ for every $j = 1$ to ℓ. Then there is no $f^{**} \in C(\mathbf{T}) \cap H_N^\infty$ such that $\overset{\rightarrow}{\gamma}_j(f^{**}) \leq \overset{\rightarrow \mathrm{btm}}{\gamma_j}$ for $j = 1, 2, ..., \ell$.

We soon prove this result which will illustrate the principle behind both 3(a) and 3(b) in the Algorithm.

THEOREM 3.2. *Suppose the performance functions $\Gamma_1, \cdots, \Gamma_\ell$ satisfy the Standard Assumption. Suppose that $f, F \in H_N^\infty$ are continuous with F never vanishing. Set $\mathbf{N}_\theta := e^{-i\theta}\bar{F}(e^{i\theta})$. Suppose that for every θ and $z \in \mathbf{N}_\theta^{r\perp}(f(e^{i\theta}))$, there exists a $j = 1, 2, ..., \ell$ such that*

$$\Gamma_j(e^{i\theta}, z) > \overset{\rightarrow \mathrm{btm}}{\gamma_j}. \tag{3.4}$$

Then there is no $f^{**} \in C(\mathbf{T}) \cap H^\infty_N$ *such that* $\overrightarrow{\gamma}_j\,(f^{**}) \leq \overrightarrow{\gamma}^{\text{btm}}_j$ *for* $j = 1, 2, ..., \ell$.

Corollary 3.3. *If* f^* *is a local optimum and* F *is its analytic dual, and if* f^*, F *are continuous, then* f^*, F *satisfy the hypotheses of Theorem 3.1 and so its conclusion gives the bound on* $\overrightarrow{\gamma}_j\,(f^*)$ *found in Theorem 3.1.*

Proof of Theorem 3.2:

The approach is similar to that which will be used in the proof of the uniqueness theorems in Section 5. Consider the transformation of $\mathbf{C}^N$ to $\mathbf{C}$ defined by $\pi_\theta(z) = e^{i\theta}F(e^{i\theta}) \cdot (z - f(e^{i\theta}))$. From assumption (3.4) we find that $\mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$ does not meet $\mathbf{N}^{r\perp}_\theta$. Thus $\operatorname{Re}\pi_\theta(z)$ is nonzero for all θ and all $z \in \mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$. Thus $\operatorname{Re}\pi_\theta(z)$ has the same sign regardless of the values of θ or $z \in \mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$; by possibly negating F, we may assume that $\pi_\theta(z)$ has strictly negative real part for all $z \in \mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$ and all θ.

Suppose $f^{**} \in H^\infty_N \cap C(\mathbf{T})$ satisfies

$$\Gamma_j(e^{i\theta}, f^{**}(e^{i\theta})) \leq \overrightarrow{\gamma}^{\text{btm}}_j \tag{3.5}$$

for every j and θ. The mapping

$$P : \mathbf{T} \to \mathbf{C}$$

$$e^{i\theta} \to \pi_\theta(f^{**}(e^{i\theta}))$$

extends to the analytic function

$$P(s) = sF(s) \cdot (f^{**}(s) - f(s))$$

for s on the closed disk and has a zero at the origin. But since $\pi_\theta(z)$ has strictly negative real part for $z \in \mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$ and all θ, the function P is nonvanishing and has strictly negative real part on the circle. Hence P has strictly negative real part at the origin. This is a contradiction. Thus f^{**} does not exist and the proof is finished.

••

That P has strictly negative real part on the circle implies that the winding number of P on the circle is zero, while $P(0) = 0$ implies that that winding number is at least 1. This contradiction foreshadows the winding number properties that we shall use in the proof of the Algorithm as stated in Theorem 3.1. The hypothesis involving q could probably be improved.

Proof of Theorem 3.1:

Suppose that there exists f^{**} as indicated in the theorem. We construct $\pi_\theta(s)$ and $P(s) = sF(s)(f^{**}(s) - f(s))$ as in the proof of Theorem 3.2. Now assumption (3.3) and the fact that F is nonzero on the circle imply that for all $\theta, 0 \notin \pi_\theta(\mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}}))$. By continuity of F and f, this implies that for some $\delta > 0$, $\pi_\theta(\mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}}))$ excludes the closed disk of radius δ about 0 in $\mathbf{C}$ for all θ. By uniform contractibility of the $\mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$ there exists a homotopy f^{**}_t from f^{**} to q such that $f^{**}_t(e^{i\theta})$ is contained in $\mathcal{S}_\theta(\overrightarrow{\gamma}^{\text{btm}})$ for every θ, t. Thus $P_t(e^{i\theta}) := \pi_\theta(f^{**}_t(e^{i\theta}))$ vanishes at no θ or t, and since it is continuous in t, θ, the winding number $\operatorname{wind}_0(P_t)$ of P_t around 0 is independent of t. Indeed, the winding number of $P(e^{i\theta})$ over the circle is the same as the winding number of $e^{i\theta}F(e^{i\theta}) \cdot (q(e^{i\theta}) - f(e^{i\theta}))$ over the circle, which is zero since we assumed that $\mathrm{Re}\,(e^{i\theta}F(e^{i\theta}) \cdot (q(e^{i\theta}) - f(e^{i\theta}))) < 0$ for all θ. Now we turn to the completely different property of P, namely $P(0) = 0$, to see that $\mathrm{wind}_0(P) > 0$. This contradicts our finding above that $\mathrm{wind}_0(P) = 0$, and thereby shows that f^{**} cannot exist.

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To state rigorous results on uniqueness requires technical definitions and we do this in the next section.

4. Uniqueness Theorem.

In this section we present the precise theorems which establish the validity of the algorithm in Section 3.

4.1. Uniqueness Theorem Expressed Analytically.

First we state a consequence of our main uniqueness theorem which is easier to understand and carries many of the main ideas.

THEOREM 4.1. *Suppose the performance functions* $\Gamma_1, \cdots, \Gamma_\ell$ *satisfy the Standard Assumption. Suppose that*

(1.) $f^* \in H^\infty_N(\mathbf{T}) \cap C^1(\mathbf{T})$ *and* $\psi_j \geq 0$ *are functions satisfying the Flatness, Gradient Alignment, Normalization and Positivity conditions, as well as the condition that* $\frac{\partial \Gamma}{\partial z}\frac{\partial \Gamma^T}{\partial z}$ *be invertible.*

(2.) Set $\mathbf{N}_\theta := e^{-i\theta}\bar{F}$. *For each* θ

$$\Gamma(e^{i\theta}, z) := \max\{\frac{\Gamma_1(e^{i\theta}, z)}{\gamma_1^*}, \cdots, \frac{\Gamma_\ell(e^{i\theta}, z)}{\gamma_\ell^*}\}$$

as a function of z *has a unique minimum for* $z \in \mathbf{N}_\theta^{r\perp}(f^*(e^{i\theta}))$ *occurring at* $z_\theta = f^*(e^{i\theta})$. *Here* $\gamma_j^* := \gamma_j(f^*)$ *denotes the performance level produced by* f^* *with respect to* Γ_j, *for* $j = 1, ..., \ell$.

Then f^* *is a unique Pareto optimum for MultiOPT at performance level* $\vec{\gamma}$. *Namely, there is no other function* $f \in H^\infty_N \cap C^1$, *with the property* $\gamma_j(f) \leq \gamma_j(f^*)$ *for all* $j = 1, \ldots, \ell$.

For the remainder of this work, we shall write $\mathbf{N}_\theta^{r\perp}, \mathbf{N}_\theta^{c\perp}$ to mean $\mathbf{N}_\theta^{r\perp}(f^*(e^{i\theta})), \mathbf{N}_\theta^{c\perp}(f^*(e^{i\theta}))$. A stronger theorem (which is considerably harder to prove) requires minimizing Γ on the set $\mathbf{N}_\theta^{c\perp}$ rather than the set $\mathbf{N}_\theta^{r\perp}$ which is one real dimension bigger than $\mathbf{N}_\theta^{c\perp}$. For SISO systems, that is $N = 1$, it is this stronger theorem and the observation $\mathbf{N}_\theta^{c\perp} = \{0\}$ which leads to the fact, mentioned in the abstract, that for SISO systems H^∞ optima are unique. Before stating the result we introduce a technical condition.

We say that a real valued function P on an affine subspace $A \subset \mathbf{C}^N$ has a *nondegenerate* (local) *minimum* at $w_0 \in A$ if P grows at least quadratically near w_0; [9] more precisely, there exists a positive C such that for all w in some neighborhood of w_0 in A, we have

$$|P(w) - P(w_0)| \geq C|w - w_0|^2.$$

THEOREM 4.2. *Suppose the performance functions* $\Gamma_1, \cdots, \Gamma_\ell$ *and* f^* *and* F *are as in the set up of Theorem 4.1 and satisfy the hypotheses (1). Replace hypothesis (2) of Theorem 4.1 by the weaker hypotheses*

(2′ a.) Set $\mathbf{N}_\theta := e^{-i\theta}\bar{F}$. *For each* θ

$$\Gamma(e^{i\theta}, z) := \max\{\frac{\Gamma_1(e^{i\theta}, z)}{\gamma_1^*}, \cdots, \frac{\Gamma_\ell(e^{i\theta}, z)}{\gamma_\ell^*}\}$$

---

[9] For our results we could replace "quadratically" with "polynomially" here; some modification would be required in Lemmas 5.1 and 5.2 to prove this.

as a function of z has a unique minimum for $z \in \mathbf{N}_\theta^{c\perp}$ occurring at $z_\theta = f^*(e^{i\theta})$. Here γ_j^* denotes the performance level produced by f^* with respect to Γ_j, for $j = 1, ..., \ell$.

($2'$ b.) For every θ, the real valued function $\Gamma(e^{i\theta}, \cdot)$ on $\mathbf{N}_\theta^{c\perp}$ given in ($2'$ a) has a nondegenerate minimum on $\mathbf{N}_\theta^{c\perp}$ at $z = f^*(e^{i\theta})$.

(Note that F is never zero on the circle. Condition ($2'$ b.) will be replaced later by condition ($2''$ b.).)

Then f^* is a unique Pareto optimum for MultiOPT at performance level $\vec{\gamma} = \vec{\gamma}\,(f^*)$. Namely, there is no other function $f \in H_N^\infty \cap C^1$, with the property $\gamma_j(f) \leq \gamma_j(f^*)$ for all $j = 1, \ldots, \ell$.

These theorems clearly give a test for determining if a local optimum is the unique global optimum for a MultiOPT problem which is practical to the extent that computing the minimum of the $\Gamma_j(e^{i\theta}, \cdot)$ over subspace $(e^{-i\theta}\bar{F})^{c\perp}$, or respectively over $(e^{-i\theta}\bar{F})^{r\perp}$, is practical. At least this is a $2N-2$ real dimensional problem, respectively $2N-1$ dimensional problem, as opposed to the infinite dimensional MultiOPT problem.

We emphasize that this condition is much less stringent than a global convexity condition that would be required for uniqueness in most optimization problems (ones not involving stability). This will be explained fully in Section 4.3 which describes our results geometrically and compares them to conventional convexity.

4.2. Uniqueness Theorem Expressed Geometrically. All of the results of this paper can be stated geometrically. This way of looking at these optimization problems strongly enhances intuition, and also geometry plays a role in our proofs (see Section 5.) Critical to a geometric understanding are the sublevel sets

$$\mathcal{S}_\theta^j(\gamma_j) := \{z \in \mathbf{C}^N : \Gamma_j(e^{i\theta}, z) \leq \gamma_j\}$$

in $\mathbf{C}^N$ of the performance functions Γ_j. Fix $\vec{\gamma}^* := (\gamma_1^*, \cdots, \gamma_\ell^*)$. Let $\partial\mathcal{S}_\theta$ denote the topological boundary of $\mathcal{S}_\theta$. Let $d\partial\mathcal{S}_\theta$ denote

$$d\partial\mathcal{S}_\theta(\gamma^*) := \partial\mathcal{S}_\theta^1(\gamma_1^*) \cap \ldots \cap \partial\mathcal{S}_\theta^\ell(\gamma_\ell^*) \qquad \forall e^{i\theta} \in \mathbf{T} \tag{4.1}$$

Of course $d\partial\mathcal{S}_\theta \subset \partial\mathcal{S}_\theta$.

The **Flatness Hypothesis** corresponds to the geometric statement
$f^*(e^{i\theta}) \in \partial\mathcal{S}_\theta^j$ whenever $\psi_j(e^{i\theta}) \neq 0$.

Hypothesis (2) of Theorem 4.1 corresponds to the geometric statement:
$\mathcal{S}_\theta$ intersects $(e^{-i\theta}\bar{F})^{r\perp}$, a "tangent" plane to $\partial\mathcal{S}_\theta$ at $f^*(e^{i\theta})$, only at $f^*(e^{i\theta})$.

Hypothesis ($2'$) of Theorem 4.2 corresponds to the geometric statement
$\mathcal{S}_\theta$ intersects $(e^{-i\theta}\bar{F})^{c\perp}$,
a complex tangent plane to $\partial\mathcal{S}_\theta$ at $f^*(e^{i\theta})$,
only at $f^*(e^{i\theta})$ and has second order contact there.

To make these last two statements comprehensible we need some definitions and also we do need to prove the statements. Tangent planes to a smooth surface can be defined as the set of points orthogonal to a normal to the surface; there are two notions of orthogonal, real and complex, which lead to two notions of tangent plane,

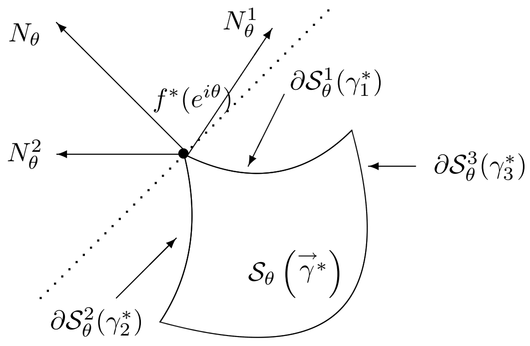

FIG. 4.1. $\mathbf{N}_\theta^j = \frac{\partial\Gamma_j}{\partial\bar{z}}(e^{i\theta}, f^*(e^{i\theta}))$ *and* $\mathbf{N}_\theta = \psi_1(e^{i\theta})\mathbf{N}_\theta^1 + \psi_2(e^{i\theta})\mathbf{N}_\theta^2 + \psi_3(e^{i\theta})\mathbf{N}_\theta^3$

the ordinary tangent plane and the complex tangent. (See below the discussion on tangents.) We are dealing with surfaces which possibly have corners. Then at a corner there are many normal directions and as a consequence many tangent planes. To get formulas for tangent planes to $\partial\mathcal{S}_\theta$ we need some background.

Consider a once continuously differentiable function ρ from $\mathbf{C}^N$ to $\mathbf{R}^+$. Let $S = \{z : \rho(z) \leq 1\}$ and $\partial S = \{z : \rho(z) = 1\}$ denote its boundary; ρ is called a defining function for ∂S. The surface ∂S is a hypersurface, that is, it has real codimension 1. The gradient $\nabla\rho$ is directed normally to ∂S at z_0 which in complex notation is

$$\nabla\rho(z_0) = \frac{\partial\rho(z_0)}{\partial\bar{z}}$$

Thus if $z_0 \in \partial\mathcal{S}_\theta^j$, then

$$\frac{\partial\Gamma_j}{\partial\bar{z}}(e^{i\theta}, z_0) \tag{4.2}$$

is directed normally to $\partial\mathcal{S}_\theta^j$. At a corner of $\mathcal{S}_\theta$ there is a family of normals $\mathcal{N}_\theta$ pointing "out" of $\mathcal{S}_\theta$ which we define to be all vectors of the form

$$\mathcal{N}_\theta := \{\psi_1(e^{i\theta})\frac{\partial\Gamma_1}{\partial\bar{z}}(e^{i\theta}, z_0) + \cdots + \psi_\ell(e^{i\theta})\frac{\partial\Gamma_\ell}{\partial\bar{z}}(e^{i\theta}, z_0) : \text{ for some } \psi_j \geq 0\} \tag{4.3}$$

This leads to a formal definition of the *tangent plane* to $\partial\mathcal{S}_\theta$ at $f^*(e^{i\theta})$ as $\mathbf{N}_\theta^{r\perp}$ for some $\mathbf{N}_\theta \in \mathcal{N}_\theta$, and the *complex tangent plane* as $\mathbf{N}_\theta^{c\perp}$ for some $\mathbf{N}_\theta \in \mathcal{N}_\theta$. (See Figure 4.) The **Gradient Alignment** condition says precisely that

$$e^{-i\theta}\bar{F}(e^{i\theta}) \in \mathcal{N}_\theta.$$

Thus the **Gradient Alignment** condition amounts to selecting a normal and corresponding tangent planes to $\mathcal{S}_\theta$ at $f^*(e^{i\theta})$. Note the nature of the corner of $\mathcal{S}_\theta$ is determined by which ψ_j are not 0 (called the active ψ_j).

To prove the geometric interpretation of Hypotheses (2) and (2′), observe that $\mathcal{S}_\theta(\overrightarrow{\gamma}) = \{z : \Gamma(e^{i\theta}, z) \leq 1\}$. Thus the Hypothesis (2) condition of Theorem 4.1

$$\Gamma(e^{i\theta}, z) > \min_{\zeta\in T}\Gamma(e^{i\theta}, \zeta) = \Gamma(e^{i\theta}, z_\theta) = 1 \text{ for } z \neq z_\theta \text{ in a set } T,$$

says that T touches $\mathcal{S}_\theta(\vec{\gamma})$ only at z_θ where $\Gamma(e^{i\theta}, z_\theta) = 1$. Thus $z_\theta = f^*(e^{i\theta})$ being the location of a unique minimum is equivalent to T touching $\mathcal{S}_\theta(\vec{\gamma})$ only at $f^*(e^{i\theta})$.

The geometrical interpretation of the lower bound presented in Section 3 is a variation on what we have just presented which is so straightforward that we will not discuss it in detail.

4.3. Benefits of Our Uniqueness Test and Comparisons. The geometric interpretations of this section lead immediately to the statement of our main result given in the Introduction, Section 1.4. Recall the striking point is that the test for uniqueness in Theorems 4.2 and 4.1 just requires checking whether a (complex) tangent plane at one point $f^*(e^{i\theta})$ per θ intersects $\mathcal{S}_\theta(\vec{\gamma}^*)$ in other points.

By contrast, convexity requires a test at all points. We now present an analogous "convexity" condition for MultiOPT, a condition which one might try to check apriori; the reader can note how much stronger the hypothesis is than the key condition $(2')$ presented in Theorem 4.2.

Theorem 4.3. *Suppose the* $\Gamma_j \quad j = 1, \cdots, \ell$ *satisfy the Standard Assumption. If for every* $\theta \in \mathbf{T}$ *and every* γ_j *all complex tangent planes to* ∂S_θ *touch* S_θ *in exactly one point and have no more than quadratic order of contact with* ∂S_θ*,* [10] *then any local optimum is a global optimum.*

Proof: For one smooth Γ this was proved by A. Vityaev [19] and independently by M. Whittlesey [20],[21]. Multiple Γ_j produce sublevel sets with corners and the slight variation of their proofs required here will be presented in the course of proving Theorem 4.2. Then Theorem 4.3 follows from Theorems 2.1 and 4.2. When "complex tangent plane" is replaced by "tangent plane" thereby producing a stronger assumption in the theorem we have conventional strict convexity; that uniqueness theorem is due to Helton and Howe [6].

4.4. Hypoconvex corners. In this section we are concerned only with uniqueness of optimum and we consider a weaker form of hypothesis $(2')$ of Theorem 4.2. It is motivated by the fact that often a differentiable local Pareto optimum must be hyperflat; see [9] for exact hypotheses guaranteeing this. Hyperflatness means that the performance of a particular optimum for every performance Γ_j is flat, i.e. $\Gamma_j(e^{i\theta}, f^*(e^{i\theta}))$ is constant as a function of θ for every $j = 1, 2, ..., \ell$. The weaker hypothesis is

$(2'''a)$ Set $\mathbf{N}_\theta := e^{-i\theta}\overline{F}$. The set $\mathbf{N}_\theta^{c\perp}$ intersects $d\partial\mathcal{S}_\theta(\vec{\gamma}^*)$ only at $f^*(e^{i\theta})$ if at all .

$(2'''b)$ There is a homotopy I_t of $d\partial\mathcal{S}_\theta(\vec{\gamma}^*)$ to $f^*(e^{i\theta})$ with $I_t(d\partial\mathcal{S}_\theta(\vec{\gamma}^*))$ lying entirely inside $\mathcal{S}_\theta(\vec{\gamma}^*)$ and missing $\mathbf{N}_\theta^{c\perp}$ for all $t,\ 0 \leq t \leq 1$.

$(2'''c)$ For every θ, the real valued function on $\mathbf{N}_\theta^{c\perp}$ given by $z \mapsto \Gamma(e^{i\theta}, z)$ has a nondegenerate minimum on $\mathbf{N}_\theta^{c\perp}$ at $z = f^*(e^{i\theta})$ (Note that this is the same as conditions $(2'b)$ and $(2''b)$ of Theorem 4.2.)

Theorem 4.4. *Assume the hypotheses of Theorem 4.2 except replace hypothesis* $(2')$ *by* $(2''')$*. Then if* f^* *and* f^{**} *are local Pareto optima both with the same performance level* $\vec{\gamma}^*$ *and if* f^{**} *is hyperflat, then* $f^{**} = f^*$.

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[10] This property is called *strict hypoconvexity* and is weaker than strict convexity because complex tangent planes are smaller than tangent planes

This theorem suggests a class of geometric objects which generalizes the notion of hypoconvexity. We say that $\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ has **hypoconvex corners** provided that at any point p of $d\partial\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ any tangent plane T_p to $d\partial\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ intersects $d\partial\mathcal{S}_\theta(\overrightarrow{\gamma}^*)$ only at p and satisfies the order of contact condition $(2'''c)$ at p. Also we require that there be a continuous homotopy I_t satisfying

$$I_t(d\partial\mathcal{S}_\theta(\overrightarrow{\gamma}^*)) \subset \mathcal{S}_\theta(\overrightarrow{\gamma}^*) \qquad 0 \le t \le 1$$

and

$$I_t(d\partial\mathcal{S}_\theta(\overrightarrow{\gamma}^*)) \cap T_p = \{p\}.$$

The appeal of having such a geometric condition is as follows. If we have a problem whose sublevel sets $\mathcal{S}_\theta(\overrightarrow{\gamma})$ all can be verified to have hypoconvex corners, then clearly no two distinct hyperflat optima f^* and f^{**} can have the same performance levels. Thus if hypoconvexity can be verified in advance, we know (even without any computer runs) that a type of uniqueness must hold.

5. Proofs of Theorem 4.1, Theorem 4.2 and Theorem 4.4. The proof of each theorem begins in the same way and they all follow the pattern laid out in the proofs in Section 3. Suppose that $f^* \in H^\infty$, F and performance levels $\gamma_1^*, \cdots \gamma_\ell^*$ exist meeting hypotheses (1) of Theorem 4.1.

Again consider the transformation

$$\pi_\theta(z) := e^{i\theta} F(e^{i\theta}) \cdot (z - f^*(e^{i\theta})).$$

Transform the sublevel sets $\mathcal{S}_\theta^j(\gamma_j^*)$ with π_θ to obtain sets

$$\tilde{\mathcal{S}}_\theta^j(\gamma_j^*) := \pi_\theta(\mathcal{S}_\theta^j(\gamma_j^*)) \subset \mathbf{C}$$

This map collapses $\mathcal{S}_\theta^j(\gamma_j^*)$ to $\mathbf{C}$ in a way which makes $\tilde{\mathcal{S}}_\theta^j(\gamma_j^*)$ contain zero .

Suppose f^{**} is an optimizer in $C(\mathbf{T})$ different from f^*. The mapping $P : \mathbf{T} \to \mathbf{C}$ defined by $e^{i\theta} \to \pi_\theta(f^{**}(e^{i\theta}))$ extends to the analytic function

$$P(s) = sF(s) \cdot (f^{**}(s) - f^*(s))$$

for s on the closed disk and has a zero at the origin. P is not identically 0, since the only point $v \in \mathcal{S}_\theta(\overrightarrow{\gamma})$ such that $\pi_\theta(v) = 0$ is $f^*(e^{i\theta})$, and for some θ, $f^*(e^{i\theta}) \neq f^{**}(e^{i\theta})$. Thus there is a $\tau > 0$ such that

(P vs. τ) $\quad [0, \tau]$ is in the image of P applied to the unit disk

(by the open mapping theorem applied to P at 0).

5.1. Proof of Theorem 4.1.

If $z \in \mathcal{S}_\theta(\overrightarrow{\gamma})$, then Assumption (2) of Theorem 4.1 implies that

$$\operatorname{Re} \pi_\theta(z) \le 0.$$

To see this, use the geometric interpretation of Assumption (2) given in Section 4.1 which says

$$\operatorname{Re} \pi_\theta(z) = \operatorname{Re}(e^{i\theta} F(e^{i\theta}) \cdot [z - f^*(e^{i\theta})]) \neq 0$$

for any $z \in \mathcal{S}_\theta(\overrightarrow{\gamma})$ except $z = f^*(e^{i\theta})$. Since $e^{-i\theta}\overline{F}(e^{i\theta})$ is the outward pointing normal we get $\operatorname{Re} \pi_\theta(z) \le 0$.

Inequality 5.1 implies $\operatorname{Re} P(e^{i\theta}) \le 0$. Moreover, $P(s)$ is analytic and bounded for s in the unit disk, since f^*, f^{**}, F are. Thus $\operatorname{Re} P(s) \le 0$ on the unit disk, and in particular $P(s) \neq \tau$ at any $|s| \le 1$. This contradicts (P vs τ). ●●

5.2. Proof of Theorem 4.2. In the course of the proof we shall need that fact that condition ($2'$ b.) implies the following condition:

($2''$ b.) There exist constants $C > 0$, $\delta > 0$ such that for all θ and $z \in \mathcal{S}_\theta(\overrightarrow{\gamma})$ such that $|z - f^*(e^{i\theta})| < \delta$, we have $|F(e^{i\theta}) \cdot (z - f^*(e^{i\theta}))| \geq C|z - f^*(e^{i\theta})|^2$.

We prove this in Lemma 5.1.

Now we must replace the $\operatorname{Re} P \leq 0$ assumption with other weaker structure. This uses the winding number (denoted wind_0) of $(P(e^{i\theta}) - \tau)$ around 0.

The first and greatest difficulty is establishing that it exists. This is accomplished by Lemma 5.2 below, which says we may choose τ small enough that τ is not in $\pi_\theta(\mathcal{S}_\theta(\overrightarrow{\gamma}))$ for any θ. Let us assume that this winding number exists and complete the proof.

For perspective note that $\operatorname{Re} P \leq 0$ implies $\operatorname{Re}[P(e^{i\theta}) - \tau] \leq -\tau$, and so $\text{wind}_0[P(e^{i\theta}) - \tau] = 0$. However, even without $\operatorname{Re} P \leq 0$ we can obtain this easily by constructing a homotopy. Recall two functions between which there exists a homotopy not passing through 0 have the same winding number about 0. Begin by constructing a homotopy f_t^{**} of f^{**} to f^* such that every $f_t^{**}(e^{i\theta})$ is contained in $\mathcal{S}_\theta(\overrightarrow{\gamma})$. We may do this by using the map I_t from the Standard Assumption: The maps $I_t(e^{i\theta}, f^*(e^{i\theta}))$ and $I_t(e^{i\theta}, f^{**}(e^{i\theta}))$ construct homotopies of f^* and f^{**} to the same continuous function so the combination is the desired homotopy. Note we do not require the functions f_t^{**} be analytic, but we merely require they be continuous. Then the functions P_t

$$e^{i\theta} \mapsto P_t(e^{i\theta}) := \pi_\theta(f_t^{**}(e^{i\theta}))$$

are a homotopy from $P_1 = P$ to $P_0 = 0$. Also,

$$0 = \text{wind}_0\ (-\tau) = \text{wind}_0\ (P_0(e^{i\theta}) - \tau) = \ \text{wind}_0\ (P(e^{i\theta}) - \tau),$$

since $P_0 = 0$. But from (P vs. τ) the analytic function $P(s) - \tau$ does equal zero for some s in the open disk, so its winding number about 0 is ≥ 1. This is a contradiction, so f^{**} cannot exist. ●●

LEMMA 5.1. *Condition ($2'$ b) implies condition ($2''$ b).*

Proof of Lemma 5.1:

Since $z \mapsto \Gamma(e^{i\theta}, z)$ has a nondegenerate minimum on $\mathbf{N}_\theta^{c\perp}$ at $z = f^*(e^{i\theta})$, there exist $\delta(\theta) > 0, C(\theta) > 0$ such that for all $z \in \mathbf{N}_\theta^{c\perp}$ such that $|z - f^*(e^{i\theta})| \leq \delta(\theta)$ we have

$$\Gamma(e^{i\theta}, z) - 1 \geq C(\theta)|z - f^*(e^{i\theta})|^2. \tag{5.1}$$

By continuity of the functions involved and a compactness argument we may assume that C and δ are independent of θ. Write $\delta(\theta) = \delta$ and $C(\theta) = C$. Without loss of generality assume that $\delta < 1$. Now suppose that the lemma does not hold; that for each $\epsilon, \gamma > 0$ there exists θ and a $z^1 \in \mathcal{S}_\theta(\overrightarrow{\gamma})$ such that $|z^1 - f^*(e^{i\theta})| < \gamma$ but

$$|F(e^{i\theta}) \cdot (z^1 - f^*(e^{i\theta}))| < \epsilon|z^1 - f^*(e^{i\theta})|^2. \tag{5.2}$$

Let z^2 equal the orthogonal projection of z^1 to $\mathbf{N}_\theta^{c\perp}$. Without loss of generality assume $\gamma < \delta$. Then the distance from z^1 to $\mathbf{N}_\theta^{c\perp}$ is $|z^1 - z^2| = |\frac{F(e^{i\theta})}{|F(e^{i\theta})|} \cdot (z^1 - f^*(e^{i\theta}))| \leq \frac{\epsilon}{|F(e^{i\theta})|}|z^1 - f^*(e^{i\theta})|^2 \leq C_F\epsilon|z^1 - f^*(e^{i\theta})|^2$, where C_F is the reciprocal of the minimum modulus of F on the circle.

Since Γ is uniformly Lipschitz on the set $\{(e^{i\theta}, z) : |z - f^*(e^{i\theta})| \leq 1\}$ and $\gamma < \delta < 1$, we find that

$$|z^1 - z^2| \leq C_F \epsilon |z^1 - f^*(e^{i\theta})|^2 \tag{5.3}$$

implies

$$\Gamma(e^{i\theta}, z^2) - 1 \leq \Gamma(e^{i\theta}, z^2) - \Gamma(e^{i\theta}, z^1) \leq |\Gamma(e^{i\theta}, z^2) - \Gamma(e^{i\theta}, z^1)| \leq C_\Gamma C_F \epsilon |z^1 - f^*(e^{i\theta})|^2, \tag{5.4}$$

where C_Γ depends only on Γ and $|\Gamma(e^{i\theta}, z^2) - \Gamma(e^{i\theta}, z^1)| \leq C_\Gamma |z^1 - z^2|$. (Recall $\Gamma(e^{i\theta}, z^1) \leq 1$ since $z^1 \in \mathcal{S}_\theta(\overrightarrow{\gamma})$.) From (5.3) and the fact that $|z^1 - f^*(e^{i\theta})| \leq \delta < 1$, we obtain $|z^1 - z^2| \leq C_F \epsilon |z^1 - f^*(e^{i\theta})|$. Now suppose we choose ϵ so small that $C_F \epsilon < \sqrt{3}/2$. Then the Pythagorean Theorem guarantees that $|z^1 - z^2|^2 + |z^2 - f^*(e^{i\theta})|^2 = |z^1 - f^*(e^{i\theta})|^2$, so

$$|z^1 - f^*(e^{i\theta})|^2 \leq 4|z^2 - f^*(e^{i\theta})|^2. \tag{5.5}$$

Combining (5.4) and (5.5),

$$\Gamma(e^{i\theta}, z^2) - 1 \leq C_\Gamma C_F 4\epsilon |z^2 - f^*(e^{i\theta})|^2,$$

which contradicts (5.1) (since $C_\Gamma C_F 4\epsilon$ can be made arbitrarily small), where we recall that $C(\theta) = C$, $z^2 \in \mathbf{N}_\theta^{c\perp}$ and $|z^2 - f^*(e^{i\theta})| \leq |z^1 - f^*(e^{i\theta})| < \gamma < \delta$.●●

Define

$$\tilde{\mathcal{S}}_\theta := \pi_\theta(\mathcal{S}_\theta).$$

We owe the reader

LEMMA 5.2.

The set $\tilde{\mathcal{S}}_\theta$ excludes the set

$$\mathcal{K} := \{\tilde{z} \in \mathbf{C} : \ \operatorname{Re} \tilde{z} > \frac{1}{2}\sqrt{|\tilde{z}|}\} \ \cap \ \{\tilde{z} \in \mathbf{C} : \ \ |\tilde{z}| < \varepsilon\}$$

where ε is some positive constant.

The set $\mathcal{K}$ is a "solid cusp" of uniform size (independent of θ) whose interior lies outside of $\tilde{\mathcal{S}}_\theta$ for all θ, and whose singularity touches every $\tilde{\mathcal{S}}_\theta$ at 0.

Proof of Lemma 5.2:

The proof splits in two parts. First we show that a $\tilde{z}$ in the image under π_θ of a $z \in \mathcal{S}_\theta$ near $f^*(e^{i\theta})$ lies in the set

$$\{\tilde{z} \in \mathbf{C} : \ \operatorname{Re} \tilde{z} \leq \frac{1}{2}\sqrt{|\tilde{z}|}\}.$$

In the second part we show that a $\tilde{z}$ in the image of a $z \in \mathcal{S}_\theta$ far from $f^*(e^{i\theta})$ satisfies $|\tilde{z}| \geq \varepsilon$.

Claim. There exists a $\delta > 0$, such that for all

$$\tilde{z} \in \pi_\theta(\ \mathcal{S}_\theta \cap \{z \in \mathbf{C}^n \quad : \ |z - f^*(e^{i\theta})| < \delta\} \ )$$

we have $\operatorname{Re} \tilde{z} \leq \frac{1}{2}\sqrt{|\tilde{z}|}$.

Proof of claim. Choose ε so small that $\forall e^{i\theta} \in \mathbf{T}$, we have $(\sum_{j=1}^{\ell} \psi_j(e^{i\theta}))\varepsilon/\sqrt{C} < 1/2$, where C comes from Assumption $2''b$. Choose ε even smaller and then δ so small that Assumption $2''b$ is satisfied and for $|z - f^*(e^{i\theta})| < \delta$, $z \in \mathcal{S}_\theta$ and $1 \leq j \leq \ell$,

$$\operatorname{Re} (\frac{\partial \Gamma_j}{\partial z}(e^{i\theta}, f^*(e^{i\theta})) \cdot (z - f^*(e^{i\theta}))) < \varepsilon |z - f^*(e^{i\theta})|.$$

Then

$$\text{Re}\,(e^{i\theta}F(e^{i\theta})\cdot(z-f^*(e^{i\theta}))) = \text{Re}\left(\sum_{j=1}^{\ell}\left(\psi_j(e^{i\theta})\frac{\partial\Gamma_j}{\partial z}(e^{i\theta},f^*(e^{i\theta}))\right)\cdot(z-f^*(e^{i\theta}))\right) \tag{5.6}$$

$$\le (\sum_{j=1}^{\ell}\psi_j(e^{i\theta}))\varepsilon|z-f^*(e^{i\theta})|. \tag{5.7}$$

By Assumption $2''b$, $|z-f^*(e^{i\theta})| \le \frac{1}{\sqrt{C}}\sqrt{|F(e^{i\theta})\cdot(z-f^*(e^{i\theta}))|}$ so

$$\text{Re}\,(e^{i\theta}F(e^{i\theta})\cdot(z-f^*(e^{i\theta}))) \tag{5.8}$$

$$\le (\sum_{j=1}^{\ell}\psi_j(e^{i\theta}))\frac{\varepsilon}{\sqrt{C}}\sqrt{|e^{i\theta}F(e^{i\theta})\cdot(z-f^*(e^{i\theta}))|} \tag{5.9}$$

$$\le \frac{1}{2}\sqrt{|e^{i\theta}F(e^{i\theta})\cdot(z-f^*(e^{i\theta}))|}, \tag{5.10}$$

i.e.

$$\text{Re}\,(\pi_\theta(z)) \le \frac{1}{2}\sqrt{|\pi_\theta(z)|}$$

for all $(s,z)\in\mathcal{S}\cap\{(s,z)\in\mathbf{T}\times\mathbf{C}^n;\ |z-f^*(e^{i\theta})|<\delta\}$. The claim follows.

For the second part of the proof, we consider $\tilde{z}\in\tilde{\mathcal{S}}_\theta$ such that $\tilde{z}=\pi_\theta(z)$ and $|z-f^*(e^{i\theta})|\ge\delta$, where δ is from the first part of the proof. Assumption $(2'a)$ says that $\mathcal{S}_\theta$ misses the complex tangent plane $\mathbf{N}_\theta^{c\perp}$ for all θ (except for $f^*(e^{i\theta})$), so

$$|\pi_\theta(z)|>0$$

for $z\in\mathcal{S}_\theta$ and $z-f^*(e^{i\theta})\ne 0$. By continuity and compactness there exists $\varepsilon>0$ such that if $z\in\mathcal{S}_\theta$ and $|z-f^*(e^{i\theta})|>\delta$ we have

$$|\pi_\theta(z)|>\varepsilon>0$$

uniformly in θ, for some ε and all z,θ, so $|\tilde{z}|>\varepsilon>0$. Combining this with the claim, the proof of the lemma is complete. ●●

Proof of Theorem 4.4: The proof follows that of Theorem 4.2. The principal difference is the existence of a homotopy f_t^{**} of f^* to f^{**} such that for all θ, $\pi_\theta(f_t^{**}(e^{i\theta}))$ does not belong to the set $\mathcal{K}$. In the proof of Theorem 4.2 the homotopy of the sets $\mathcal{S}$ is the tool that provides us with this fact. Now that we know that the optimum f^* is hyperflat, its graph lies in a smaller set, so the homotopy of the entire $\mathcal{S}$ is not needed; the homotopy of the set where $\Gamma_j=\gamma_j$ for every j (which contains the graph of f^*) will suffice. The only change, then, arises in the second part of the proof of Lemma 5.2, where assumption $(2'a)$ is used. ●●

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