

RIEMANN SURFACES IN FIBERED POLYNOMIAL HULLS

MARSHALL A. WHITTLESEY

Texas A&M University

ABSTRACT. Let Δ be the closed unit disk in \mathbf{C} , let Γ be the circle, let $\Pi : \Delta \times \mathbf{C} \rightarrow \Delta$ be projection, and let $A(\Delta)$ be the algebra of complex functions continuous on Δ and analytic in $\text{int } \Delta$. Let K be a compact set in \mathbf{C}^2 such that $\Pi(K) = \Gamma$, and let $K_\lambda \equiv \{w \in \mathbf{C} \mid (\lambda, w) \in K\}$. Suppose further that (a) for every $\lambda \in \Gamma$, K_λ is the union of 2 nonempty disjoint connected compact sets with connected complement, (b) there exists a function $Q(\lambda, w) \equiv (w - R(\lambda))^2 - S(\lambda)$ quadratic in w with $R, S \in A(\Delta)$ such that for all $\lambda \in \Gamma$, $\{w \in \mathbf{C} \mid Q(\lambda, w) = 0\} \subset \text{int } K_\lambda$ where S has only one zero in $\text{int } \Delta$, counting multiplicity and (c) for every $\lambda \in \Gamma$, the map $w \mapsto Q(\lambda, w)$ is injective on each component of K_λ . Then we prove that $\widehat{K} \setminus K$ is the union of analytic disks 2-sheeted over $\text{int } \Delta$, where \widehat{K} is the polynomial convex hull of K . Furthermore, we show that $\partial \widehat{K} \setminus K$ is the disjoint union of such disks.

Let Δ be the closed unit disk in \mathbf{C} , let Γ be the circle and let $\Pi : \Delta \times \mathbf{C} \rightarrow \Delta$ be projection. Let K be a compact set such that $\Pi(K) = \Gamma$. Numerous authors (see [1],[5],[6],[8],[9],[12]) have studied features of the polynomial hull of K , denoted by \widehat{K} , frequently to investigate whether \widehat{K} contains analytic structure in the form of graphs of analytic functions whose boundaries land in K . (Such functions are commonly called analytic selectors for K .) In this endeavour, it is natural to restrict oneself to the case where the fiber of K over $\lambda \in \Gamma$, $K_\lambda \equiv \{w \in \mathbf{C} \mid (\lambda, w) \in K\}$ is a connected compact set with connected complement (so also polynomially convex). (See [5],[6],[9].)

We now consider the case of a compact K where the fibers are not necessarily connected, but still have connected complements (and so are still polynomially convex.) We shall specify circumstances where the part of the polynomial hull of K which projects through Π onto $\text{int } \Delta$ is the union of analytic disks which are not graphs over $\text{int } \Delta$ but are 2-sheeted over $\text{int } \Delta$. Under the same circumstances,

1991 *Mathematics Subject Classification*. Primary 32E20; Secondary 30D50.

Key words and phrases. polynomial hull, analytic structure.

we shall show that $\partial\widehat{K} \setminus K$ is the disjoint union of such analytic disks. Let $A(\Delta)$ denote the disk algebra of functions continuous on Δ and analytic on $\text{int } \Delta$, and let $H^\infty(\Delta)$ denote the algebra of bounded analytic functions on $\text{int } \Delta$. We consider K with the following properties:

- (1) $\left[\begin{array}{l} \text{(a) For every } \lambda \in \Gamma, K_\lambda \text{ is the union of 2 nonempty disjoint connected} \\ \text{compact sets with connected complement;} \\ \text{(b) There exists a function } Q(\lambda, w) \equiv (w - R(\lambda))^2 - S(\lambda) \text{ quadratic in } w \\ \text{with } R, S \in A(\Delta) \text{ such that for all } \lambda \in \Gamma, \{w \in \mathbf{C} \mid Q(\lambda, w) = 0\} \subset \text{int } K_\lambda \\ \text{where } S \text{ has only one zero in } \text{int } \Delta, \text{ counting multiplicity;} \\ \text{(c) For every } \lambda \in \Gamma, \text{ the map } w \mapsto Q(\lambda, w) \text{ is injective on each component} \\ \text{of } K_\lambda. \end{array} \right.$

Note that (c) implies that S has no zeroes on Γ and that the points in $\{w \in \mathbf{C} \mid Q(\lambda, w) = 0\}$ lie in different components of K_λ , $\lambda \in \Gamma$. Property (c) is easily obtained if, for example, the diameters of the components of K_λ are sufficiently small.

We shall prove the following.

Theorem 1. *If K is a compact set satisfying (1) then $\widehat{K} \setminus K$ is the union of the interiors of analytic disks of the form*

$$(2) \quad \begin{aligned} \text{int } \Delta &\longrightarrow \widehat{K} \\ \Gamma &\longrightarrow K \text{ for a.e. } \lambda \in \Gamma \\ z &\longmapsto (B(z), f(z)) \end{aligned}$$

where B is a Blaschke product of order 2 and $f \in H^\infty(\Delta)$

(so the accumulation points on the boundary of the disk land in K).

First we prove a theorem which allows more components in the fibers of K_λ but requires a relation among the components.

Theorem 2. *Let M and Y be compact sets fibered over the circle (i.e., $\Pi(M) = \Pi(Y) = \Gamma$) such that $\widehat{M} \neq M$ and Y has fibers $Y_\lambda \subset \mathbf{C}$, $\lambda \in \Gamma$, which are connected with connected complement. Suppose that there exists a function*

$$Q(\lambda, w) = \sum_{n=0}^d a_n(\lambda) w^n$$

with $a_n \in A(\Delta)$ for all n and $a_d \equiv 1$ such that for all $\lambda \in \Gamma$,

$$M_\lambda = \{w \in \mathbf{C} \mid Q(\lambda, w) \in Y_\lambda\}.$$

Then $\widehat{M} \setminus M$ is the union of analytic varieties d -sheeted over $\text{int } \Delta$.

Proof. Let $(\lambda_0, w_0) \in \widehat{M} \setminus M$. Then we claim that $(\lambda_0, Q(\lambda_0, w_0)) \in \widehat{Y} \setminus Y$:
Given a polynomial P ,

$$\begin{aligned} |P(\lambda_0, Q(\lambda_0, w_0))| &\leq \sup_{(\lambda, w) \in M} |P(\lambda, Q(\lambda, w))| \\ &\leq \sup_{\{(\lambda, w) \mid (\lambda, Q(\lambda, w)) \in Y\}} |P(\lambda, Q(\lambda, w))| \\ &\leq \sup_{(\lambda, w) \in Y} |P(\lambda, w)| \end{aligned}$$

as claimed.

Since for $\lambda \in \Gamma$ the Y_λ are connected with connected complement, there exists $f \in H^\infty(\Delta)$ such that

$$f(\lambda_0) = Q(\lambda_0, w_0)$$

and the accumulation points of the graph of f over Γ land in Y . Then we have that

$$\{(\lambda, w) \in \text{int } \Delta \times \mathbf{C} \mid Q(\lambda, w) = f(\lambda), |\lambda| < 1\}$$

is an analytic variety passing through (λ_0, w_0) whose accumulation points over Γ land in M . \square

Corollary 1. *If M and Y are as in Theorem 2 then*

$$\{(\lambda, w) \in \widehat{M} \setminus M\} = \{(\lambda, w) \in \text{int } \Delta \times \mathbf{C} \mid (\lambda, Q(\lambda, w)) \in \widehat{Y} \setminus Y\}.$$

Proof. The inclusion \subset was proven in the theorem. As for the opposite: take (λ_0, w_0) with $(\lambda_0, Q(\lambda_0, w_0)) \in \widehat{Y} \setminus Y$. Then there exists an $f \in H^\infty(\Delta)$ such that $f(\lambda_0) = Q(\lambda_0, w_0)$ and such that the set of accumulation points of the graph of f over Γ is contained in Y . Then

$$\{(\lambda, w) \in \text{int } \Delta \times \mathbf{C} \mid (\lambda, Q(\lambda, w)) \in \text{the graph of } f \text{ over } \text{int } \Delta\}$$

is an analytic variety over $\text{int } \Delta$ passing through (λ_0, w_0) with accumulation points over Γ in M . Thus $(\lambda_0, w_0) \in \widehat{M} \setminus M$, as desired. \square

Example. Suppose M is a compact set defined over Γ such that M_λ is the union of two disks of radius $1/2$, centered at $\pm\sqrt{\lambda}$. Let us take $Q(\lambda, w) = w^2$. We claim that M has the required properties described in Theorem 2. First, given a fixed $\lambda \in \Gamma$, choose a square root $\sqrt{\lambda}$. Then the image of $\{w \in \mathbf{C} \mid |w - \sqrt{\lambda}| \leq \frac{1}{2}\}$ under the map $w \rightarrow w^2$ is the same as the image of $\{w \in \mathbf{C} \mid |w + \sqrt{\lambda}| \leq \frac{1}{2}\}$. We call the image Y_λ ; since the squaring map is 2-1, M_λ is the preimage of Y_λ under the squaring map. Letting Y be the set with fibers Y_λ , we see that Y is compact. Also Y has connected and simply connected fibers because the squaring map is 1-1 in a neighborhood of each of the components of M_λ so is a homeomorphism from each component to Y_λ . Hence $\widehat{M} \setminus M$ is the union of varieties of the form $w^2 = f(\lambda)$ where $f \in H^\infty(\Delta)$ and $f(\lambda) \in Y_\lambda$ for a.e. $\lambda \in \Gamma$.

Next we require two lemmas.

Lemma 1. *If U and V are in $A(\Delta)$ and V has exactly one zero in Δ (not on Γ), then $\{(\lambda, w) \in \Delta \times \mathbf{C} \mid (w - U(\lambda))^2 - V(\lambda) = 0\}$ is a 2-sheeted analytic disk over Δ whose boundary is a continuous closed curve.*

Proof. We may write

$$V(\lambda) = \frac{\lambda - \alpha}{1 - \bar{\alpha}\lambda} e^{\phi(\lambda)}$$

where $\phi \in A(\Delta)$, $|\alpha| < 1$. Then our surface over Δ is

$$\left\{ (\lambda, w) \in \Delta \times \mathbf{C} \mid \left(\frac{w - U(\lambda)}{e^{\frac{\phi(\lambda)}{2}}} \right)^2 - \frac{\lambda - \alpha}{1 - \bar{\alpha}\lambda} = 0 \right\}$$

which, via the change of coordinates

$$(\lambda', w') = \left(\frac{\lambda - \alpha}{1 - \bar{\alpha}\lambda}, \frac{w - U(\lambda)}{e^{\frac{\phi(\lambda)}{2}}} \right)$$

biholomorphic in $\text{int } \Delta \times \mathbf{C}$ and continuous in $\Delta \times \mathbf{C}$, is equivalent to

$$\{(\lambda', w') \in \Delta \times \mathbf{C} \mid (w')^2 - \lambda' = 0\},$$

a 2-sheeted disk. \square

Lemma 2. *If $U, V \in A(\Delta)$ and for all $\lambda \in \Gamma$, the solutions of $(w - U(\lambda))^2 - V(\lambda) = 0$ lie in K_λ (one in each component) then V has exactly one zero in Δ , counting multiplicity, which is not on Γ .*

Proof. Choose ϵ small enough that if $\lambda \in \Gamma$, the components of K_λ are at least 3ϵ apart in distance. From Lemma 1 and the remark following (1), we conclude that the analytic variety in (1) given by $\{(\lambda, w) \in \text{int } \Delta \times \mathbf{C} \mid (w - R(\lambda))^2 - S(\lambda) = 0\}$ is an analytic disk 2-sheeted over $\text{int } \Delta$. Suppose it is parametrized with $z \mapsto (B(z), g(z))$, $|z| \leq 1$. Then B is analytic in $\text{int } \Delta$ and maps the closed disk $2 - 1$ onto itself. Clearly $B \in A(\Delta)$ (using the transformation from Lemma 1), and maps Γ to Γ . Thus B is a Blaschke product of order 2. Now the solutions of $(w - R(\lambda))^2 - S(\lambda) = 0$ over λ are $R(\lambda) \pm \sqrt{S(\lambda)}$, where $\sqrt{S(\lambda)}$ is not well defined over Γ . However, since $S \circ B$ has winding number 2 over Γ , $\sqrt{S \circ B}$ can be continuously well defined over Γ ; we choose it so that $R(B(z)) + \sqrt{S \circ B}(z)$ equals $g(z)$. Then we choose $\sqrt{V \circ B}$ so that $U(B(z)) + \sqrt{V \circ B}(z)$ lies in the same component of $K_{B(z)}$ as $g(z)$. Construct a path $p(z, t)$ from $g(z)$ to $U(B(z)) + \sqrt{V \circ B}(z)$ which varies continuously in (z, t) and always stays within ϵ of $K_{B(z)}$. Then we find through the homotopy p that $\text{wind}(2\sqrt{V \circ B}) = \text{wind}(U \circ B + \sqrt{V \circ B} - (U \circ B - \sqrt{V \circ B})) = \text{wind}(R \circ B + \sqrt{S \circ B} - (R \circ B - \sqrt{S \circ B})) = \text{wind}(2\sqrt{S \circ B}) = 1$, so $\text{wind}(V \circ B) = 2$ and hence the winding number of V is one over Γ . Thus V has exactly one zero on Δ , since it has none on Γ (the roots of $(w - U(\lambda))^2 - V(\lambda) = 0$ are distinct for $\lambda \in \Gamma$). \square

In order to distinguish between elements of the copy of Δ that we began with and elements of the domain of functions such as B and g above which parametrize the 2-sheeted disks, we generally use λ to refer to the elements of the former and z to refer to elements of the latter.

Combining Lemmas 1 and 2, we see that given any continuously bounded analytic variety $\{(\lambda, w) \in \Delta \times \mathbf{C} \mid (w - U(\lambda))^2 - V(\lambda) = 0\}$ with $U, V \in A(\Delta)$ over Δ where the fiber of the variety over λ has one point in each component of K_λ , it must be a 2-sheeted analytic disk with boundary over Δ .

In order to prove Theorem 1, we shall first assume that K is a smoothly bounded solid torus: i.e., we shall assume that there exists a mapping

$$\begin{aligned} \mathcal{I} : \Gamma \times \Gamma &\rightarrow \Gamma \times \mathbf{C} \\ (z, w) &\longmapsto (z^2, I(z, w)) \end{aligned}$$

such that the following hold, where K is the compact set whose fibers over $\lambda \in \Gamma$ are $\widehat{I(z, \Gamma)} \cup \widehat{I(-z, \Gamma)}$, for $z^2 = \lambda$:

$$(3) \left[\begin{array}{l} \text{(a) } I \text{ is of class } C^2; \\ \text{(b) } \frac{\partial I}{\partial w}(z, w) \text{ is never } 0; \\ \text{(c) for any } z \in \Gamma, I(z, \cdot) \text{ is injective.} \end{array} \right.$$

We shall need the fact that there exists a compact set M as in Theorem 2, also satisfying (1), such that $K_\lambda \subset M_\lambda$ for all $\lambda \in \Gamma$. To see this, let X denote the compact set whose fiber X_λ is $\{w \in \mathbf{C} \mid Q(\lambda, w) = Q(\lambda, w') \text{ for some } w' \in K_\lambda\}$. In other words, $X_\lambda = K_\lambda \cup (2R(\lambda) - K_\lambda)$, where $2R(\lambda) - K_\lambda = \{w \in \mathbf{C} \mid w = 2R(\lambda) - w' \text{ for some } w' \in K_\lambda\}$. Then we claim that X_λ consists of two connected components. Let $K_{\lambda,1}$ and $K_{\lambda,2}$ denote the components of K_λ and let $K'_{\lambda,1}$ and $K'_{\lambda,2}$ denote their reflections $2R(\lambda) - K_{\lambda,1}$ and $2R(\lambda) - K_{\lambda,2}$ in $R(\lambda)$, respectively. Then $X_\lambda = K_{\lambda,1} \cup K_{\lambda,2} \cup K'_{\lambda,1} \cup K'_{\lambda,2}$. Clearly $K_{\lambda,1} \cap K'_{\lambda,2} \neq \emptyset$ and $K'_{\lambda,1} \cap K_{\lambda,2} \neq \emptyset$. Also $K_{\lambda,1} \cup K'_{\lambda,2}$ doesn't meet $K'_{\lambda,1} \cup K_{\lambda,2}$ because (i) $K_{\lambda,1} \cap K_{\lambda,2} = \emptyset$ and $K'_{\lambda,1} \cap K'_{\lambda,2} = \emptyset$ from (1)(a) and (ii) $K_{\lambda,1} \cap K'_{\lambda,1} = \emptyset$ and $K_{\lambda,2} \cap K'_{\lambda,2} = \emptyset$ from (1)(c). This establishes the claim. Since the components of X_λ are symmetric about $R(\lambda)$, the polynomial hulls of the components are as well, and are disjoint because the two components of X_λ are connected. Thus if we define X' over Γ to have fibers \widehat{X}_λ and M to be the closure of X' in $\Gamma \times \mathbf{C}$ then M satisfies (1) and the properties that M does in Theorem 2, and $M \supset K$.

We shall need (3) when invoking results from [5],[9] and [10].

Let w_1 be one of the elements of \mathbf{C} such that $Q(1, w_1) = 0$. Then in fact we will show that, with the additional conditions (3), $\widehat{K} \setminus K$ is the union of analytic disks

of the form (2) where

$$(4) \quad \begin{aligned} B(z) &= e^{i\theta} z \frac{z - \alpha}{1 - \bar{\alpha}z}, B(1) = 1, |\alpha| \leq 1 - \epsilon \text{ for some } \epsilon > 0, \\ f &\in A(\Delta), \text{ and } f(1) \text{ is in the same component of } K_1 \text{ as } w_1. \end{aligned}$$

We shall also need the fact that K can be continuously expanded to a solid torus slightly larger than M . In other words, we construct mappings $\mathcal{I}_t(z, w) = (z^2, I_t(z, w))$, $0 \leq t \leq 2$ having the same properties as \mathcal{I} above in (3) and let K^t be the compact set whose fibers over $\lambda \in \Gamma$ are $I_t(\widehat{\sqrt{\lambda}}, \Gamma) \cup I_t(-\widehat{\sqrt{\lambda}}, \Gamma)$. We require that $K_\lambda^{t_1} \subset \text{int } K_\lambda^{t_2}$ if $t_1 < t_2$, $K^t = \bigcap_{s>t} K^s$, $K^0 = K$, $M_\lambda \subset \text{int } K_\lambda^1$ for all $\lambda \in \Gamma$ and for all t , $0 \leq t \leq 2$ K^t satisfies the properties that K does in (1). To do this, we follow a method of Ślodkowski [9,p.371]. Suppose that we first construct a compact N satisfying the same properties K does in (1) and (3), and $M_\lambda \subset \text{int } N_\lambda$ for all $\lambda \in \Gamma$. We may also construct N so that the associated map \mathcal{I}_N extends to be a diffeomorphism of the interior of the solid torus by extending each $I_N(z, \cdot)$ from Γ to Δ . (We leave the verification of this intuitively obvious fact to the reader.) By composing the inverse of \mathcal{I}_N with the diffeomorphism $(z, w) \rightarrow (z, w/(1 - |w|^2))$, we can map the sets K, M to sets K', M' in Ślodkowski's setting in $\Gamma \times \mathbf{C}$, construct the associated $(K^t)'$ there, and pull them back through the above diffeomorphism to obtain the K^t . The only difference now is that Ślodkowski only needed $(K^1)'$ large enough to contain the graph of a constant function. By using a compactness argument, we can extend this so that $(K^1)'$ contains any particular compact set in $\Gamma \times \mathbf{C}$, say M' , so that K^1 contains M . The remaining properties are easily verified.

Lemma 3. *There exists an $\epsilon > 0$ such that if $B(z) = e^{i\theta} z \frac{z - \alpha}{1 - \bar{\alpha}z}$ and the mapping $z \mapsto (B(z), g(z))$ is an analytic disk continuous for $z \in \Delta$ with boundary in K^1 then $|\alpha| < 1 - \epsilon$.*

Proof. Suppose that for a sequence of continuously bounded analytic disks with boundary in K^1 2-sheeted over $\text{int } \Delta$ we obtain $B_n, g_n \in A(\Delta)$ parametrizing them as above, with associated α_n tending to 1 in modulus, and $\theta_n \rightarrow \theta$. Then on compact subsets of $\text{int } \Delta$, $B_n(z)$ converges to $e^{i\phi} z$ (for some real constant ϕ) and

$g_n \rightarrow g$. Thus $\widehat{K}^1 \setminus K^1$ contains an analytic graph over $\text{int } \Delta$. If we restrict the corresponding function to the region $|\lambda| < 1 - \delta$ where δ is chosen so small that for all λ on the circle of radius $1 - \delta$, $\widehat{K}^1_\lambda \subset K^2_\lambda$, (possible since $K^1 \subset \text{int } K^2$ in $\Gamma \times \mathbf{C}$) then we have a continuous selector for the set K^2 over $|\lambda| = 1$. The topology of K^2 does not permit this. Thus the possible α must have modulus bounded above by $1 - \epsilon$ for some $\epsilon > 0$. \square

Letting ϵ be as found in Lemma 3, let \widetilde{K}^t equal the union of K^t with the union over $\text{int } \Delta$ of all analytic disks possessing properties (2) and (4), replacing K by K^t .

Theorem 3. *Let K be a compact set fibered over Γ satisfying properties (1) and suppose there exist functions $\mathcal{I}(z, w) = (z^2, I(z, w))$ satisfying properties (3). Then $\widehat{K} \setminus K$ is the union of the interiors of analytic disks of the form*

$$\begin{aligned} \Delta &\longrightarrow \widehat{K} \\ \Gamma &\longrightarrow K \\ z &\longmapsto (B(z), f(z)) \end{aligned}$$

where B is a Blaschke product of order 2, $f \in A(\Delta)$.

Proof. If (B, f) is a pair satisfying (2) and (4), replacing K by K^t , let $\sqrt{S \circ B}$ denote the continuous square root of $S \circ B$ over Γ such that $R(1) + \sqrt{S \circ B}(1)$ is in the same component of $K^t_{B(1)}$ as w_1 . (Note that the winding number of $S \circ B$ is 2 on Γ .) Let

$$\begin{aligned} L^t(B) = \{(z, w) \in \Gamma \times \mathbf{C} \mid (B(z), w) \in K^t, w \in \text{the same component of } K^t_{B(z)} \\ \text{as } R(B(z)) + \sqrt{S \circ B}(z)\} \end{aligned}$$

and let

$$\widetilde{K}^t(B) = K^t \cup \{(\lambda, w) \in \text{int } \Delta \times \mathbf{C} \mid (\lambda, w) = (B(z), w) \text{ where } (z, w) \text{ lies on the}$$

graph of some element of $A(\Delta)$ which is an analytic selector for $L^t(B)\}$.

Then \widetilde{K}^t is the union of all sets $\widetilde{K}^t(B)$ ranging over all possible Blaschke products B satisfying (4). Now let $s =$ the infimum of all t such that $\widetilde{K}^t \supset \widehat{K}$. We first show that $s \leq 1$ and eventually $s = 0$. We apply Theorem 2 to M and obtain

the corresponding set Y with connected fibers specified in Theorem 2: $Y_\lambda \equiv \{w \in \mathbf{C} \mid w = Q(\lambda, w') \text{ for some } w' \in M_\lambda\}$. Following Ślodkowski [9, p.380] we write Y as the decreasing intersection of sets Y^n fibered over Γ whose boundaries are smooth tori; write $M_\lambda^n = \{w \in \mathbf{C} \mid Q(\lambda, w) \in Y_\lambda^n\}$. For some large n , $M^n \subset K^1$. Now Theorem 3 of [5], Theorem 1.1 of [9] and Theorem 4 of [10] show that $\widehat{Y}^n \setminus Y^n$ is the union of graphs over $\text{int } \Delta$ of elements of $A(\Delta)$. Then Corollary 1 shows that $\widehat{M}^n \setminus M^n$ is the union of varieties of the form $\{(\lambda, w) \in \text{int } \Delta \times \mathbf{C} \mid Q(\lambda, w) = f(\lambda), f \in A(\Delta)\}$ with boundary in M^n . Lemmas 1 and 2 shows that such a variety must be an analytic disk; in particular, suppose this disk is parametrized by $z \mapsto (B(z), g(z))$, $|z| \leq 1$. Then as in Lemma 2, we find that B is a Blaschke product of order 2 and $g \in A(\Delta)$. By change of coordinates in z we may assume that $B(0) = 0$, and $(B(1), g(1)) = (1, v_1)$, where v_1 is in the same component of M_λ as w_1 . Since $M^n \subset K^1$, this shows that indeed $\widetilde{K}^1 \supset \widetilde{M}^n = \widehat{M}^n \supset \widehat{M} \supset \widehat{K}$, as desired. Thus $s \leq 1$.

We want to prove that $s = 0$, so we make the following

(5) Assumption: $s > 0$.

Claim: $\widetilde{K}^s \supset \widehat{K}$.

Take $(\lambda, w) \in \widehat{K}$. Then clearly $(\lambda, w) \in \widetilde{K}^s$ if $|\lambda| = 1$. If $|\lambda| < 1$, then for $n \geq 1$ take $\{B_n\}, \{f_n\}$ possessing properties (2),(4) (replacing K by $K^{s+\frac{1}{n}}$) such that $(\lambda, w) \in \widetilde{K}^{s+\frac{1}{n}}(B_n)$. Then $\exists z_n \in \text{int } \Delta$, $f_n \in A(\Delta)$ which is an analytic selector for $L^{s+\frac{1}{n}}(B_n)$ with $(B_n(z_n), f_n(z_n)) = (\lambda, w)$. If $B_n(z) = e^{i\theta_n} z \frac{z - \alpha_n}{1 - \overline{\alpha}_n z}$ then without loss of generality we may assume that $\alpha_n \rightarrow \alpha_0$, $|\alpha_0| \leq 1 - \epsilon$, $\theta_n \rightarrow \theta_0$, $z_n \rightarrow z_0$, $|z_0| < 1$, (if $|z_0| = 1$ then since $B_n \rightarrow B_0$ uniformly, $|B_n(z_n) - B_0(z_0)| \leq |B_n(z_n) - B_0(z_n)| + |B_0(z_n) - B_0(z_0)|$ tends to zero as $n \rightarrow \infty$ so $1 > |\lambda| = |B_n(z_n)| \rightarrow |B_0(z_0)| = 1$, which is impossible) and $f_n \rightarrow f_0$ uniformly on compact subsets of $\text{int } \Delta$. Also note that we have chosen $\sqrt{S \circ B_n}, \sqrt{S \circ B_0}$ such that $\sqrt{S \circ B_n}(z)$ converges to $\sqrt{S \circ B_0}(z)$ for all $z \in \Gamma$.

Subclaim: $(z_0, f_0(z_0)) \in \widehat{L}^s(B_0)$, where $B_0(z) = e^{i\theta_0} z \frac{z - \alpha_0}{1 - \overline{\alpha}_0 z}$.

We have $(z_n, f_n(z_n)) \in \widehat{L^{s+\frac{1}{n}}(B_n)}$ for $n \geq 1$. Fix polynomial $P(z, w)$, fix $\epsilon > 0$, let

$$C = \sup_{(z,w) \in L^s(B_0)} |P(z, w)|$$

and take N_1 so large that

$$L^{s+\frac{1}{N_1}}(B_0) \subset \{(z, w) \in \Delta \times \mathbf{C} \mid |P(z, w)| < C + \epsilon\},$$

(using the fact that $K^s = \bigcap_{t>s} K^t$, so $L^s(B_0) = \bigcap_{t>s} L^t(B_0)$) and choose $N_2 \geq N_1$ so large that for $n \geq N_2$

$$L^{s+\frac{1}{N_1}}(B_n) \subset \{(z, w) \in \Delta \times \mathbf{C} \mid |P(z, w)| < C + \epsilon\}.$$

Then for $n > N_2$,

$$L^{s+\frac{1}{n}}(B_n) \subset \{(z, w) \in \Delta \times \mathbf{C} \mid |P(z, w)| < C + \epsilon\}.$$

and choose N_2 even larger so that for $n > N_2$,

$$|P(z_n, f_n(z_n)) - P(z_0, f_0(z_0))| \leq \epsilon,$$

possible since $z_n \rightarrow z_0$, $|z_0| < 1$ and $f_n \rightarrow f_0$ uniformly on compact subsets of $\text{int } \Delta$. Then $|P(z_0, f_0(z_0))| \leq \sup_{L^s(B_0)} |P| + 2\epsilon$, and this holds for any $\epsilon > 0$, so

$$|P(z_0, f_0(z_0))| \leq \sup_{L^s(B_0)} |P|$$

and hence $(z_0, f_0(z_0)) \in \widehat{L^s(B_0)}$. This proves the subclaim.

Take $f \in A(\Delta)$ which is an analytic selector for $L^s(B_0)$ (see Theorem 3 of [5], Theorem 1.1 of [9] and Theorem 4 of [10]) and whose graph passes through $(z_0, f_0(z_0))$. This shows that $(B_0(z_0), f(z_0)) = (\lambda, w) \in \widetilde{K^s}$, as desired. Hence $\widetilde{K^s} \supset \widehat{K}$, which was our claim.

We now claim that $\widehat{K} \setminus \bigcup_{t<s} \widetilde{K^t}$ is nonempty. Relative to the topology of $\Delta \times \mathbf{C}$, $\widetilde{K^{t_1}}$ contains a neighborhood of $\widetilde{K^{t_2}}$ if $t_1 > t_2$ since $L^{t_1}(B)$ contains a neighborhood of $L^{t_2}(B)$ in $\Gamma \times \mathbf{C}$ for any B . Thus for any r , $\bigcup_{t<r} \widetilde{K^t}$ is relatively open in $\Delta \times \mathbf{C}$.

Thus if $\bigcup_{t < s} \widetilde{K}^t$ contains \widehat{K} , then for some $r < s$, \widetilde{K}^r contains a neighborhood of \widehat{K} in $\Delta \times \mathbf{C}$. (This holds because the interiors of the \widetilde{K}^t in $\Delta \times \mathbf{C}$ form an open cover of \widehat{K} , and \widehat{K} is compact.) This contradicts the minimality of s .

Thus there exists some $p = (B(z_0), f(z_0)) \in \widehat{K} \setminus \bigcup_{t < s} \widetilde{K}^t$. Clearly $|z_0| < 1$. Then $(z_0, f(z_0)) \in \widehat{L^s(B)} \setminus \bigcup_{t < s} \widehat{L^t(B)}$.

We claim that this means $f(z) \in \partial L_z^s(B) \forall z \in \Gamma$. To see this, suppose that at some point $\zeta \in \Gamma$, $f(\zeta) \in \text{int } L_\zeta^s(B)$. By continuity of f , this holds in a neighborhood of ζ in Γ . Now let $N(z)$ be the inward pointing unit normal to $\partial L_z^s(B)$ at $f(z)$, if $f(z) \in \partial L_z^s(B)$. Choose a polynomial $G(z)$ such that $\arg G$ is within $\frac{\pi}{10}$ of $\arg N(z) / \left(\frac{z-z_0}{1-\bar{z}_0 z} \right)$ where $N(z)$ is defined, and arbitrary elsewhere on Γ except that $G(z) \neq 0$ on Γ and $\text{wind } G$ equals 0. If we let $F(z) = G(z) \left(\frac{z-z_0}{1-\bar{z}_0 z} \right)$ then $F \in A(\Delta)$, $\arg F$ is within $\frac{\pi}{10}$ (modulo 2π) of $\arg N(z)$ where $N(z)$ is defined, F is never zero on Γ and $F(z_0) = 0$. Hence for sufficiently small positive τ , $f(z) + \tau F(z) \in \text{int } L_z^s(B)$ for all z in Γ . (This is obvious pointwise for $z \in \Gamma$ and can be extended to the entire circle uniformly in τ by a compactness argument.) Furthermore, the graph of $f + \tau F$ passes through $(z_0, f(z_0))$. This contradicts the minimality of s and we conclude $f(z) \in \partial L_z^s(B) \forall z \in \Gamma$.

We consider the various possibilities for the value of the winding number of $(f - R \circ B - \sqrt{S \circ B})$ over Γ . We may show through an argument like the above that if the winding number were positive, s would not be minimal. We next show that this winding number is either 0 or -1 .

If $\text{wind}(f(z) - R(B(z)) - \sqrt{S \circ B}(z)) = d < 0$ then $\text{wind}(f(z) - R(B(z)) - \sqrt{S \circ B}(z))(f - R(B(z)) + \sqrt{S \circ B}(z)) = 1 + d = \text{wind}((f(z) - R(B(z)))^2 - S(B(z)))$ which is ≥ 0 since $(f(z) - R(B(z)))^2 - S(B(z))$ is analytic, and nonzero on Γ since $s > 0$. Hence $d = -1$.

Case 1. $\text{Wind}(f - R \circ B - \sqrt{S \circ B}) = 0$.

Let $Q(\lambda, w) = (w - U(\lambda))^2 - V(\lambda)$ be analytic in $\text{int } \Delta \times \mathbf{C}$, continuous on $\Delta \times \mathbf{C}$, and zero on points $(B(z), f(z))$, $z \in \Delta$.

In the proof of Lemma 1 we find a change of coordinates in $\Delta \times \mathbf{C}$ which

is analytic in $\text{int } \Delta \times \mathbf{C}$ and which carries $Q(\lambda, w)$ to $w^2 - \lambda$. Let us switch to these coordinates, obtaining sets J^t as the image of the sets K^t . Under this transformation we observe that \widetilde{K}^t maps to \widetilde{J}^t and \widehat{K} to \widehat{J} . Then s satisfies the same extremal property with respect to the J^t as the K^t . Now in the new coordinates the J^t are not as smooth as the K^t but needed properties will be preserved. In particular, (i) the winding number above is the same since the function $e^{\frac{\phi(\lambda)}{2}}$ in Lemma 1 has no zeroes in Δ and (ii) for fixed t the fibers of J^t are still smoothly bounded and have uniformly bounded curvature since $e^{\frac{\phi(\lambda)}{2}}$ is bounded away from zero. Now with our change of coordinates we find that B is transformed into the squaring map and f into the identity. We write $p = (w_0^2, w_0)$.

Let $n(w)$ = the inward unit normal to $J_{w_0^2}^s$ at w and let $N(w) = 2wn(w)$ be the image of $n(w)$ under the differential of $w \mapsto w^2 - \lambda$. (Note that $n(w)$ is still continuous under the change of coordinates.) Then $\text{wind}(N(w)) = 1$. Choose a polynomial g such that $|\arg g(w) - \arg N(w)| < \frac{\pi}{10}$ (modulo 2π) and $g(w_0) = 0$. Now consider the set where

$$w^2 - \lambda = \tau g(w)$$

for some fixed small positive constant τ . We need a lemma. Let D be a closed disk in \mathbf{C} centered at 0 such that $J^2 \subset \Gamma \times \text{int } D$.

Lemma 4. *For τ sufficiently small,*

$$(6) \quad w^2 - \lambda = \tau g(w)$$

has exactly 2 solutions for w in $\text{int } D$ for all $\lambda \in \Delta$, the solutions actually lie in \widehat{J}^2_λ as well, and for $\lambda \in \Gamma$, the solutions lie in different components of \widehat{J}^2_λ .

Proof. Suppose the assertion for $\lambda \in \Delta$ does not hold. Then take $\tau_n \downarrow 0$, $\lambda_n \rightarrow \lambda \in \Delta$ such that (6) has other than two solutions for $w \in \text{int } D$, where we replace λ, τ in (6) by λ_n, τ_n . Suppose this number of solutions is equal to k_n .

Since $w^2 - \lambda_n - \tau_n g(w) \rightarrow w^2 - \lambda$ uniformly for w in a compact set in \mathbf{C} , as $n \rightarrow \infty$, $w^2 - \lambda_n - \tau_n g(w)$ has the same number of zeroes in D as $w^2 - \lambda$. So for large n , $k_n = 2$, a contradiction. The argument regarding \widehat{J}^2_λ is similar.

To prove the assertion regarding $\lambda \in \Gamma$, we may proceed by contradiction again and use a similar argument to come to the conclusion that for some λ , $w^2 - \lambda$ doesn't vanish at one of $\pm\sqrt{\lambda}$, an obvious contradiction. \square

We claim that for $\lambda \in \Gamma$, these zeroes are in fact in $\text{int } J_\lambda^s$ for small τ . Let $h(z, \tau)$ denote the location of the zero for $w^2 - z^2 - \tau g(w)$ which is in the same component of $\widehat{J^2}_{z^2}$ as z . (Note $\lambda = z^2$.) We claim that h is a C^∞ function in z, τ for sufficiently small τ . We know that h satisfies the equation

$$F(h, z, \tau) \equiv h^2 - z^2 - \tau g(h) = 0.$$

Fix v , $|v| = 1$. Then since $\frac{\partial F}{\partial h} = 2h - \tau g'(h) = 2v \neq 0$ when $(h, z, \tau) = (v, v, 0)$, the implicit function theorem shows that h is a C^∞ function of (z, τ) in a neighborhood of $(v, 0)$. Choosing finitely many such neighborhoods covering all $v \in \Gamma$ we find that indeed h has the required smoothness.

We check that the set $\{(\lambda, w) \in \Delta \times \text{int } D \mid w^2 - \lambda = \tau g(w)\}$ is in fact (for the above small τ) given by $\{(\lambda, w) \in \Delta \times \mathbf{C} \mid w^2 + a_1(\lambda)w + a_0(\lambda) = 0\}$ for some $a_1(\lambda), a_0(\lambda) \in A(\Delta)$. Let $r_\tau^1(\lambda), r_\tau^2(\lambda)$ be the two solutions, not well-defined, of (6) for $w \in \text{int } D$. Then we just have to show that $r_\tau^1 + r_\tau^2$ and $r_\tau^1 r_\tau^2$ are both elements of $A(\Delta)$. Consider the well-defined continuous function $(r_\tau^1(\lambda) - r_\tau^2(\lambda))^2$ on Δ ; near where r_τ^1 is different from r_τ^2 , $r_\tau^1(\lambda)$ and $r_\tau^2(\lambda)$ can be well-defined and are analytic; thus $(r_\tau^1(\lambda) - r_\tau^2(\lambda))^2$ is continuous and analytic on Δ where it is nonzero. By Radó's Theorem, $(r_\tau^1(\lambda) - r_\tau^2(\lambda))^2$ is in $A(\Delta)$. Thus its zeroes are isolated in $\text{int } \Delta$. We conclude that $r_\tau^1 + r_\tau^2$ and $r_\tau^1 r_\tau^2$ are analytic except at isolated points where $r_\tau^1(\lambda) = r_\tau^2(\lambda)$. But both functions are clearly bounded on Δ so such singularities are removable. Hence $r_\tau^1 + r_\tau^2$ and $r_\tau^1 r_\tau^2$ are both elements of $A(\Delta)$, as desired.

We have $h : \Gamma \times (-\delta, \delta) \longrightarrow \mathbf{C}$ for some small δ and

$$h(w, \tau)^2 - \lambda - \tau g(h(w, \tau)) = 0.$$

Differentiating implicitly with respect to τ ,

$$2h \frac{\partial h}{\partial \tau} - g(h(w, \tau)) - \tau g'(h(w, \tau)) \frac{\partial h}{\partial \tau} = 0$$

for small $|\tau|$. When $\tau = 0$,

$$2w \frac{\partial h}{\partial \tau} - g(w) = 0,$$

so

$$\frac{\partial h}{\partial \tau} = \frac{g(w)}{2w} = \frac{\frac{g(w)}{N(w)} 2wn(w)}{2w} = r(w)n(w),$$

where r is a continuous nonzero function in $w \in \Gamma$ with argument within $\frac{\pi}{10}$ of 0. This means that for small positive τ , the zeroes of $w^2 - \lambda - \tau g(w)$ in $\text{int } D$ over λ lie in the interior of J_λ^s . Deferring the verification of this for a moment, we see that from Lemmas 1,2 this means we have constructed a continuously bounded 2-sheeted analytic disk in J^t for some $t < s$. This disk passes through (w_0^2, w_0) since $g(w_0) = 0$. This is a contradiction of the minimality of s and hence Case 1 is impossible.

To check the above assertion, first choose ϵ so small that a vector pointing with argument within $\frac{\pi}{5}$ of the inward pointing normal to J_λ^s at w (where $w^2 = \lambda$) lies entirely in $\text{int } J_\lambda^s$ (except for w) if its length is less than ϵ . (Here we used the fact that the curvature of the J_λ^s in the new coordinates is bounded uniformly in λ .) Then choose δ so small that for $|\tau| < \delta$ and $z \in \Gamma$ (i) $|h(z, \tau) - h(z, 0)| < \epsilon$ and (ii) $|\arg \frac{\partial h}{\partial \tau}(z, \tau) - \arg \frac{\partial h}{\partial \tau}(z, 0)| < \frac{\pi}{10}$. Then for $0 < |\tau| < \delta$, $\arg \left(\frac{h(z, \tau) - h(z, 0)}{\tau} \right) = \arg \left(\frac{\partial h}{\partial \tau}(z, \tau_z) \right)$ for some τ_z between 0 and τ , by the mean value theorem. Hence for $0 < \tau < \delta$, $h(z, \tau) - h(z, 0)$ has length less than ϵ and has argument within $\frac{\pi}{5}$ of the inward pointing normal to $J_{z^2}^s$ at z so $h(z, \tau)$ lies in $\text{int } J_{z^2}^s$ for all $0 < \tau < \delta$ and $z \in \Gamma$.

Case 2. $\text{Wind}(f - R \circ B - \sqrt{S \circ B}) = -1$.

Let us apply the same coordinate transformation of Case 1. Let $n(w)$ and $N(w)$ be as before; then $\text{wind}(N(w)) = 0$. Choose g analytic in a neighborhood of Δ such that $\arg(g(w))$ is within $\frac{\pi}{10}$ of $\arg(-N(w))$ for $|w| = 1$ and consider the set

$$(7) \quad \{(\lambda, w) \in \Delta \times \mathbf{C} \mid w^2 - \lambda = \tau g(w)\} \cap \widehat{J^2};$$

for small τ ; using an argument similar to that in Case 1, this is an analytic 2-sheeted disk whose fiber over $\lambda \in \Gamma$ consists of 2 points outside of J_λ^s . As $\tau \downarrow 0$,

(7) approaches the point (w_0^2, w_0) . We also claim that (7) doesn't meet \widehat{J} . We show this by proving (7) doesn't meet any 2-sheeted analytic disk in $\widetilde{J^s} \setminus J^s$ for any sufficiently small τ .

Parametrize (7) by $\lambda \mapsto (B_\tau(\lambda), f_\tau(\lambda))$ which possesses property (4). Let $\sqrt{B_\tau}$ be the continuous square root of B_τ on Γ such that $\sqrt{B_\tau}(1) = 1$. Using reasoning similar to that at the end of Case 1, choose δ so small that (7) doesn't meet J^s for $0 < \tau < \delta$ and $f_\tau(w) - \sqrt{B_\tau}(w)$ has argument within $\frac{\pi}{5}$ of $\arg(-n(w))$, modulo 2π . Consider a disk in $\widetilde{J^s}$ given by $\{(\lambda, w) \in \Delta \times \mathbf{C} \mid U(\lambda, w) = 0\}$, U monic quadratic in w . Now there are 2 well defined continuous functions $R_\tau^1(z), R_\tau^2(z)$ such that the zeroes of $U(\lambda, w)$ over $\lambda = B_\tau(z)$ are $R_\tau^1(z), R_\tau^2(z)$; just let $R_\tau^1(z)$ be the zero of $U(B_\tau(z), w)$ which lies in the same component of $J_{B_\tau(z)}^1$ as $\sqrt{B_\tau}(z)$ and let $R_\tau^2(z)$ be the other zero.

Then $U(B_\tau(w), f_\tau(w)) = (f_\tau(w) - R_\tau^1(w))(f_\tau(w) - R_\tau^2(w))$. Now over $|w| = 1$,

$$\begin{aligned} \text{wind}(f_\tau(w) - R_\tau^1(w)) &= \text{wind}(f_\tau(w) - \sqrt{B_\tau}(w)) \\ &(\text{since (7) doesn't meet } J^s \text{ for } 0 < \tau < \delta) \\ &= \text{wind } n(w) \\ &(\text{since } f_\tau(w) - \sqrt{B_\tau}(w) \text{ has argument within } \frac{\pi}{5} \text{ of } \arg(-n(w))) \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} \text{wind}(f_\tau(w) - R_\tau^2(w)) &= \text{wind}(\sqrt{B_\tau}(w) - R_\tau^2(w)) \\ &= \text{wind}(\sqrt{B_\tau}(w) - (-\sqrt{B_\tau}(w))) = 1 \end{aligned}$$

so $\text{wind}(U(B_\tau(w), f_\tau(w))) = 0$; this means that U is never 0 on (7) for such τ . This holds for all U defining a 2-sheeted disk in $\widetilde{J^s}$, so for $0 < \tau < \delta$, (7) doesn't meet $\widetilde{J^s}$, so doesn't meet \widehat{J} .

Let $P(\widehat{J})$ be the set of continuous complex functions on \widehat{J} which are uniform limits of polynomials. Let $Q_\tau(\lambda, w)$ be monic quadratic in w , $0 < \tau < \delta$, such that $Q_\tau(B_\tau(w), f_\tau(w)) = 0$ for all w . Then by the Oka-Weil Theorem, $Q_\tau(\lambda, w)^{-1}$ is an element of $P(\widehat{J})$, since (7) doesn't meet \widehat{J} . Also $Q_\tau(\lambda, w)^{-1}$ is bounded on $J = J^0$ uniformly in τ , $0 < \tau < \delta$, since $s > 0$ but as $\tau \rightarrow 0$, $Q_\tau(w_0^2, w_0)^{-1} \rightarrow \infty$,

a contradiction. Thus the original assumption (5) that $s > 0$ must be false; $s = 0$ and $\tilde{K} \supset \widehat{K}$. (This concludes Case 2.)

We already know $\tilde{K} \subset \widehat{K}$, so $\tilde{K} = \widehat{K}$, as desired. \square

Proof of Theorem 1. Following Ślodkowski [9], choose compact sets $K(n)$ satisfying (1) such that $K = \bigcap K(n)$ and the $K(n)$ are solid tori whose boundaries arise from mappings $\mathcal{I}(n)$ which are restricted by (3). Also choose the $K(n)$ such that for all (λ, n) , $K_\lambda \subset \subset K(n+1)_\lambda \subset \subset K(n)_\lambda$.

We now invoke Theorem 3, replacing K by $K(n)$ and conclude that $\widehat{K(n)} = \widehat{K(n)}$, so $\widehat{K(n)} \supset \widehat{K}$. Thus $\widehat{K} \subset \widehat{K(n)}$ for all n . Thus every point p in $\widehat{K} \setminus K$ lies on a sequence of analytic disks parametrized by $z \rightarrow (B^n(z), f^n(z))$, where B^n, f^n possess properties (4) with respect to $K(n)$. If $B^n \rightarrow B$ uniformly and $f^n \rightarrow f$ uniformly on compact sets then using an argument similar to that in the Claim of the proof of Theorem 3, we can conclude that every point of the form $(B(\lambda), f(\lambda))$ for $|\lambda| < 1$ lies in $\widehat{K} \setminus K$. The associated disk contains the point p and its boundary accumulation points lie in K , as desired. (Note that this shows $\widehat{K} = \bigcap \widehat{K(n)}$.) \square

Theorem 4. *If K is as in Theorem 1, then $\partial\widehat{K} \setminus K$ is the disjoint union of 2-sheeted analytic disks.*

Proof. First suppose K has the special form in Theorem 3. Choose a point $(\lambda, w) \in \partial\widehat{K}$ where $|\lambda| < 1$, and suppose (λ, w) lies on a disk parametrized by $z \mapsto (B(z), f(z))$. The analysis in the proof of Theorem 3 shows that we can choose f to be continuous on Δ , $f(z) \in \partial L_z(B)$ for all $z \in \Gamma$ and Case 2 of the proof of Theorem 3 holds. (Otherwise we can construct $g \in A(\Delta)$ such that $g(z) \in \text{int } L_z(B)$ for all $z \in \Gamma$ and the disk parametrized by $z \mapsto (B(z), g(z))$ passes through (λ, w) . Small perturbations of g then show that $(\lambda, w) \notin \partial\widehat{K}$.) Then in Case 2 we showed that every point on the disk $z \mapsto (B(z), f(z))$ is the limit of points on 2-sheeted disks external to \widehat{K} . (Actually we proved this for \widehat{J} but the coordinate transformation allows us to pull it back to \widehat{K} .) Hence all of the disk $z \mapsto (B(z), f(z))$ lies in $\partial(\widehat{K}) \setminus K$. Thus $\partial(\widehat{K}) \setminus K$ is the union of 2-sheeted analytic disks over $\text{int } \Delta$.

To see that these disks are disjoint, suppose two of them given by $z \mapsto (B^1(z), f^1(z))$ and $z \mapsto (B^2(z), f^2(z))$ meet in $\text{int } \Delta \times \mathbf{C}$. Assume without loss of generality that $(B^1(z_1), f^1(z_1)) = (B^2(z_2), f^2(z_2))$ for some $z_1, z_2 \in \text{int } \Delta$. Now construct the sequence of disks $z \mapsto (B_\tau^1(z), f_\tau^1(z))$ external to \widehat{K} , as in Case 2 of the proof of Theorem 3. By changing coordinates in z , assume that $(B_\tau^1(z_1), f_\tau^1(z_1)) \rightarrow (B^1(z_1), f^1(z_1))$. Let $P^i(\lambda, w)$ be monic quadratic in w such that $P^i(B^i(z), f^i(z)) = 0$ for all $z \in \Gamma$, $i = 1, 2$. Then the functions $P^i(B_\tau^1(z), f_\tau^1(z))$ are nonzero analytic functions in z which tend to 0 at $z = z_1$ (as $\tau \rightarrow 0$) for $i = 1, 2$. Pass to a subsequence of (B_τ^1, f_τ^1) which converges locally uniformly (and nontrivially, without loss of generality) to $((B^1)', (f^1)')$. By Hurwitz' Theorem, $\{P^i(B_\tau^1(z), f_\tau^1(z))\}_\tau$ tends to zero uniformly for z in compact subsets of $\text{int } \Delta$ as $\tau \rightarrow 0$ and we conclude that the two disks $z \mapsto (B^1(z), f^1(z))$ and $z \mapsto (B^2(z), f^2(z))$ parametrize the same analytic disk because for every $\lambda \in \text{int } \Delta$, $P^1(\lambda, w)$ and $P^2(\lambda, w)$ vanish for the same two values of w .

For general K , write K as a decreasing intersection of $K(n)$ as before; then $\widehat{K} = \cap \widehat{K}(n)$, as noted at the end of the proof of Theorem 1. Choose $(\lambda, w) \in \partial \widehat{K} \setminus K$. Then, passing to a subsequence of the $K(n)$, there exist points $(\lambda_n, w_n) \in \partial \widehat{K}(n) \setminus K(n)$ converging to (λ, w) . With them are associated 2-sheeted disks $z \mapsto (B^n(z), f^n(z))$ in $\partial \widehat{K}(n)$ which pass through (λ_n, w_n) . A local uniform limit can be chosen as before so that $z \mapsto (B(z), f(z))$ passes through (λ, w) and lies in $\partial \widehat{K} \setminus K$. To show no two 2-sheeted disks in $\partial \widehat{K}(n) \setminus K(n)$ meet, we can employ an argument similar to that in the previous paragraph, using the (B^n, f^n) instead of the (B_τ^1, f_τ^1) . \square

The author is grateful to Professors Herbert Alexander, Brian Birgen, Brian Cole, John Wermer and the referee for their suggestions and to the Brown University Department of Mathematics for its hospitality while part of this paper was being written.

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COLLEGE STATION, TX 77843-3368, USA
E-mail address: mwhittle@math.tamu.edu