IV. The *icosahedron* ($n = 3$, $\nu = 5$, $z = 20$). We choose ten meridians 36° apart and call them 1, 2, 3, . . ., 10. On the meridians 1, 3, 5, 7, 9 we lay off from N the equal arcs NA, NB, NC, ND, NE, and on the meridians 6, 8, 10, 2, 4 we lay off from S the equal arcs SA', SB', SC', SD', SE' such that the ten triangles NAB, NBC, NCD, NDE, NEA, $SA'B'$, $SB'C'$, $SC'D'$, $SD'E'$, $SE'A'$ are equilateral. The common length $2k$ of the marked-off arcs can be obtained, for example, from one of the right triangles NBO, NCO, into which the meridian 4 divides the equilateral triangle NBC. Since $\measuredangle BNO = 36°$, $\measuredangle OBN = 72°$, it follows from triangle NBO that

$$\cos BO = \cos k = \frac{\cos 36°}{\sin 72°} = \frac{1}{2 \sin 36°}$$

and from this that $2k = 63°26'$.

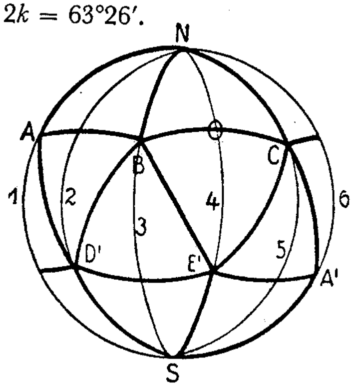

Fig. 83.

If we extend NO by its own length to H, we obtain the isosceles triangle NBH with the base $NH = 2h$ and the legs $BN = BH = 2k$, the base angle 36°, and the apex angle $HBN = 144°$. Since these angles have the same sine, the sines of their opposite sides NH and NB are equal according to the sine theorem. But since these opposite sides ($2h$ and $2k$) are not equal, $2h$ must be the supplement of $2k$. And since NE' is also the supplement of $2k$ ($= SE'$), then necessarily

$$NE' = 2h = NH.$$

Accordingly, point H coincides with E' and $E'B$ is equal to $2k$, i.e., equal to NB. In similar fashion each of the arcs AD', $D'B$, $E'C$, CA', $A'D$, DB', $B'E$, EC', $C'A$ is equal to $2k$, and the ten "encircling" triangles ABD', $D'E'B$, BCE', $E'A'C$, CDA', $A'B'D$, DEB', $B'C'E$, EAC', $C'D'A$ are likewise equilateral triangles and also congruent to the ten equilateral triangles above.

The 12 points N, S, A, B, C, D, E, A', B', C', D', E' are thus the vertexes of 20 equilateral triangles that completely cover the sphere; they are the 12 corners of the regular icosahedron.

V. The *dodecahedron* ($n = 5$, $\nu = 3$, $z = 12$). As in the icosahedron, we begin the construction of the dodecahedron by laying off a system of ten meridians 1, 2, 3, . . ., 10 that are 36° apart. About N as a common apex we group five congruent isosceles triangles NAB, NBC, NCD, NDE, NEA with the apex angle 72° and the base angle 60° ($= 180°/\nu$) whose base vertexes A, B, C, D, E lie on the meridians 1, 3, 5, 7, 9. Thus we obtain the regular pentagon $ABCDE$. In the same way we draw about S as a common center point the regular pentagon $A'B'C'D'E'$ whose vertexes A', B', C', D', E' lie on the meridians 6, 8, 10, 2, 4.

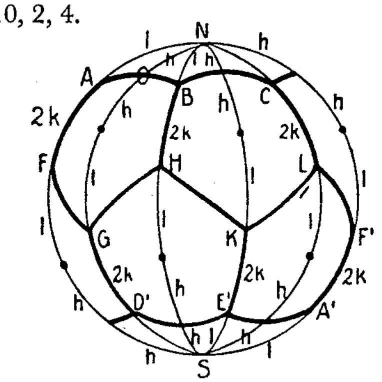

Fig. 84.

If O and O' represent the base midpoints of the isosceles triangles ABN and $D'E'S$, then NAO and $SD'O'$ are right triangles with the angles 60° and 36°.

Our construction is now based on the theorem (proved below): "*The perimeter of a spherical right triangle with angles of* 60° *and* 36° *is* 90°." If we designate the hypotenuse, the long leg, and the short leg of such a triangle as l, h, and k, then

(1) $$l + h + k = 90°.$$

If we remember that

$$NA = SD' = l, \qquad NO = SO' = h, \qquad AO = D'O' = k,$$

we see that $2k$ is the side, l the radius of the circumscribed circle (on the sphere), h the radius of the inscribed circle, and $s = l + h$ the altitude of the pentagon $ABCDE$ or $A'B'C'D'E'$.