96 A Comet Inside the Earth's Orbit

What is the maximum number of days that a comet can remain within the earth's orbit?

We will assume that the earth's orbit is circular and the comet's parabolic, and that the orbital planes coincide.

SOLUTION. We will select the large half axis of the earth's orbit as the unit length, the mean solar day as the unit time, and we will designate the parabola parameter as $4k$, the base line of the parabola section lying within the earth's orbit as $2y$, the altitude of the section as x, the sector described by the focal radius of the comet within the earth's orbit as S, and finally, the time required to traverse the sector as t. Then

$$(1) \qquad y^2 = 4kx$$

according to the amplitude equation of the parabola,

$$(2) \qquad (x - k)^2 + y^2 = 1$$

according to the circle equation, and

$$(3) \qquad 3S = y(x + 3k)$$

according to the formula for the area of a parabola section [No. 56. S = the section − triangle $= \frac{4}{3}xy - (x - k)y$].

If $2p$ represents the orbit parameter of a celestial body of mass μ revolving about the sun (the mass of the sun is considered as the unit mass), if t is any time, S the sector described by the body in this time, we can use the *Gauss formula**

$$\frac{2S}{t\sqrt{p}\sqrt{1 + \mu}} = G,$$

where G (the root of the gravitation constant) is the so-called Gauss constant, which has the numerical value of 0.0172021 for the units assumed.

Since the mass of the comet relative to that of the sun is negligible, the Gauss formula is transformed into

$$(4) \qquad S = Ct\sqrt{k}, \quad \text{with} \quad C = G/\sqrt{2}$$

in our problem.

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* Gauss, *Theoria motus corporum coelestium in sectionibus conicis solem ambientium* (Hamburg, 1809). (English translation by C. H. Davis reprinted by Dover Publications, 1963.)

From (1) and (2) we find

$$x + k = 1, \qquad y = 2\sqrt{k(1 - k)}$$

and, making use of these values, we obtain from (3)

$$3S = 2\sqrt{k(1 - k)}(1 + 2k).$$

If we introduce here the value for S from (4), it follows that

$$(5) \qquad t = c(1 + 2k)\sqrt{1 - k}, \quad \text{with} \quad c = \sqrt{8}/3G.$$

Since t is to be a maximum, the expression $(1 + 2k)\sqrt{1 - k}$ must be made as great as possible. It therefore remains to select k in such manner that the expression or its square or fourth power, namely,

$$P = (1 + 2k)\cdot(1 + 2k)\cdot(4 - 4k),$$

becomes a maximum. However, since P is a product of factors of constant sum, it attains a maximum (No. 10) when the factors are equally great, thus when

$$1 + 2k = 4 - 4k.$$

This gives us $k = \frac{1}{2}$ and, as a result of (5), $t = 78$.

The sought-for *maximum possible length of stay* is thus 78 days.

97 The Problem of the Shortest Twilight

On what day of the year is the twilight shortest at a place of given latitude?

This problem was posed, but not solved, by the Portuguese Nunes in 1542 in his book *De crepusculis*. Jacob Bernoulli and d'Alembert solved the problem by means of differential calculus, but obtained no simple results. The first elementary solution stems from Stoll (*Zeitschrift für Mathematik und Physik*, vol. XXVIII). The following very simple solution is from Brünnow's *Lehrbuch der sphärischen Astronomie* (Textbook of Spherical Astronomy).

A distinction is made between civil and astronomical twilight. Civil twilight ends when the midpoint of the sun stands $6\frac{1}{2}°$ below the horizon. Approximately at this moment one must turn on one's lights in order to continue working. Astronomical twilight ends when the midpoint of the sun stands 18° below the horizon; it is approximately at this time that the astronomer can begin making observations.

It is convenient to choose as the beginning of twilight the moment at which the midpoint of the sun is intersected by the horizon.

Let the latitude of the observation point be φ, the pole distance of the sun p.

The duration of the twilight is measured by the angle d that is formed by the two-hour circle arcs of the nautical triangles determined by the sun for the beginning and end of the twilight. If we superimpose one of these triangles on the other in such manner that the two pole distances coincide, the angle between the two latitude complements b (now having in common only the world pole P) represents

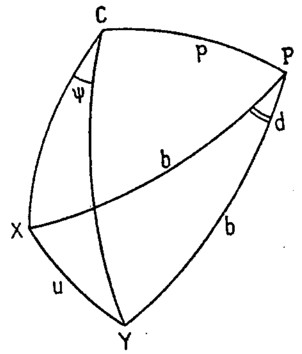

FIG. 110.

the duration d of the twilight. In this position let the triangles be PCX and PCY, with $PC = p$, $PX = PY = b = 90° - \varphi$, $CX = 90°$, $CY = 90° + h$ (h is to be understood as representing the depth of the sun below the horizon at the end of the twilight), and $\measuredangle XPY = d$. Moreover, let $XY = u$ and $\measuredangle XCY = \psi$.

From the isosceles triangle PXY it follows, according to the cosine theorem, that

$$(1) \qquad \cos d = \frac{\cos u - \sin^2 \varphi}{\cos^2 \varphi}.$$

Consequently, d becomes a minimum or $\cos d$ a maximum when $\cos u$ is at a maximum.

From the triangle CXY it follows, however, that

$$\cos u = \cos CX \cos CY + \sin CX \sin CY \cos \psi$$

or, since $\cos CX = 0$, $\sin CX = 1$, $\sin CY = \cos h$, that

$$\cos u = \cos h \cos \psi.$$

Thus, $\cos u$ attains its greatest possible value when $\cos \psi$ is a maximum, i.e., when

$$\psi = 0.$$

On the day of the shortest twilight, point X accordingly falls on the side CY, and the base $XY = u$ of the isosceles triangle PXY is h. At the same time we find from (1) for the minimum duration $\mathfrak{d}$ of the twilight

$$\cos \mathfrak{d} = \frac{\cos h - \sin^2 \varphi}{\cos^2 \varphi}$$

or, in accordance with the two formulas

$$\cos \mathfrak{d} = 1 - 2 \sin^2 \frac{\mathfrak{d}}{2}, \qquad \cos h = 1 - 2 \sin^2 \frac{h}{2},$$

$$(\mathrm{I}) \qquad \sin \frac{\mathfrak{d}}{2} = \frac{\sin \frac{h}{2}}{\cos \varphi}.$$

To find the *corresponding* declination of the sun δ, we express the cosine of the angle $\omega = \measuredangle PCX = \measuredangle PCY$ twice in accordance with the cosine theorem and set the resulting values equal to each other.

It follows from $\triangle PCX$ (since $\cos CX = 0$, $\sin CX = 1$) that

$$\cos \omega = \frac{\sin \varphi}{\sin P},$$

from $\triangle PCY$ (since $\cos CY = -\sin h$, $\sin CY = \cos h$) that

$$\cos \omega = \frac{\sin \varphi + \cos p \sin h}{\sin p \cos h}.$$

Equalizing, we obtain

$$\sin \varphi \cos h = \sin \varphi + \cos p \sin h$$

or

$$-\cos p \sin h = \sin \varphi (1 - \cos h)$$

or

$$-\cos p \cdot 2 \sin \frac{h}{2} \cos \frac{h}{2} = \sin \varphi \cdot 2 \sin^2 \frac{h}{2}$$

or, finally,

$$\cos p = -\sin \varphi \tan \frac{h}{2}.$$

Because of the minus sign, the pole distance p is an obtuse angle for northern latitudes, the sun's declination δ is thus *southerly* and

$$\text{(II)} \qquad \sin\delta = \sin\varphi \tan\frac{h}{2}.$$

The shortest twilight duration is determined by (I) and the southerly declination of the sun for the day on which that twilight occurs is given by (II).

From the declination the sought-for day can be found by means of the nautical almanac.

This datum is also found with sufficient accuracy if the familiar formula

$$(2) \qquad \sin\delta = \sin\varepsilon \sin l$$

is used; here δ represents the sun's declination, l the angular distance of the sun from the autumnal or vernal equinox, and ε the inclination of the ecliptic (23° 27′). Since the above-mentioned angular distance changes at an average daily rate of $m = 59.1'$, the sought-for information varies by $n = l/m$ days from the 23rd of September or from the 21st of March.

For Leipzig, for example, ($\varphi = 51° 20.1'$) we find, from (II), $\delta = 7° 6.2'$, then from (2), $l = 18° 6.3'$, and then $n = 18.4$. The shortest twilight in Leipzig thus falls on October 11 and March 3.

98 Steiner's Ellipse Problem

Of all the ellipses that can be circumscribed about (inscribed in) a given triangle, which one has the smallest (largest) area?

"Dans le plan, la question des polygones d'aire maximum ou minimum inscrits ou circonscrits à une ellipse ne présente aucune difficulté. Il suffit de projeter l'ellipse de telle manière qu'elle devienne un cercle, et l'on est ramené à une question bien connue de géométrie élémentaire"* (Darboux, *Principes de Géométrie analytique*, p. 287).

* Translation: "In a plane the question of polygons of maximum or minimum area inscribed in or circumscribed about an ellipse offers no difficulty. All that is necessary is to project the ellipse in such manner that it is transformed into a circle, and the problem is reduced to a well-known question of elementary geometry".

The solution of the problem is based on the two auxiliary theorems:

I. *Of all the triangles inscribed in a circle the one possessing the maximum area is the equilateral.*

II. *Of all the triangles that can be circumscribed about a circle the one possessing the minimum area is the equilateral.*

Proof of I. We call the circle diameter d, the sides and angles of an inscribed triangle p, q, r and α, β, γ, respectively, the area of the triangle J. Then

$$J = \tfrac{1}{2}pq \sin\gamma$$

and

$$p = d\sin\alpha, \qquad q = d\sin\beta,$$

and consequently,

$$J = \tfrac{1}{2}d^2 \cdot \sin\alpha \sin\beta \sin\gamma.$$

According to No. 92, the product of the sines $\sin\alpha \sin\beta \sin\gamma$ of the three angles α, β, γ of constant sum (180°) is at a maximum when

$$\alpha = \beta = \gamma (= 60°),$$

i.e., when the triangle is equilateral. The area of this maximal triangle is $\frac{3}{16}\sqrt{3}d^2$, thus $\sqrt{27}/4\pi$ of the area of the circle.

Proof of II. If we designate the sides of an arbitrary circumscribed triangle PQR as p, q, r, then the tangents to the circle from the vertexes P, Q, R are $x = s - p$, $y = s - q$, $z = s - r$, where s represents half the perimeter of the triangle

$$\left(s = \frac{p + q + r}{2} = x + y + z\right).$$

The area J of the triangle and the radius ρ of the inscribed circle are given by the well-known formulas

$$J = \rho s \quad \text{and} \quad J = \sqrt{xyzs} \quad \text{(Hero of Alexandria)}.$$

These give us

$$s\rho^2 = xyz.$$

Making use of the formula $J = \rho s$, we write this equation in the following two ways:

$$(1) \qquad \frac{1}{yz} + \frac{1}{zx} + \frac{1}{xy} = \frac{1}{\rho^2},$$

$$(2) \qquad \frac{1}{yz} \cdot \frac{1}{zx} \cdot \frac{1}{xy} = \frac{1}{J^2\rho^2}.$$