Math. 510 HW #3

① Case 1 Say φ is divisible by π. We show ψ, χ are also. Since $\varphi-\psi=(\xi-\eta)\pi$, and $\pi \mid \varphi$, π must divide ψ. Since $\pi \mid \varphi, \psi$, π must divide $\varphi+\psi=\chi$, as desired.

Case 2 Say ψ is divisible by π. Then since $\varphi=\psi+(\xi-\eta)\pi$, $\pi \mid$ the left side, so $\pi \mid \varphi$. Then since $\pi \mid \varphi, \psi$, $\pi \mid \varphi+\psi=\chi$, as desired.

Case 3 Say χ is divisible by π. Since $\varphi-\psi=(\xi-\eta)\pi$, $\varphi+\psi=\chi$, $2\varphi=(\xi-\eta)\pi+\chi$. So $\pi \mid$ left side $\Rightarrow$ $\pi \mid 2\varphi$. Since π and 2 are prime, and $\pi \nmid 2$, we have that $\pi \mid \varphi$. Then from Case 1, π also divides ψ.

In the text, top of p. 99: if we divide ξ, η by 2, why does the remainder ϵ satisfy $\epsilon^3=\pm 1$.

Property VI, p. 101 says $\xi=2\lambda+\epsilon$, $\eta=2\mu+\epsilon$ where $N(\epsilon) \le \frac{3}{4}N(2)=3$. So by III, p100, $N(\epsilon)=1$ or 3. But if $N(\epsilon)=3$, that means that at the bottom of p.101, $|r|=|s|=\frac{1}{2}$, so, since $\beta=2$ here, $\rho=\pm\frac{1}{2}J\pm\frac{1}{2}\Theta$, so $\epsilon=\sigma=\pm J\pm\Theta$. So $\sigma=\pm 1$ or $\pm\pi$.

In the latter case, $\pi=2J-1$, $-\pi=-2J+1$, so in fact we can get remainder when dividing by 2 down to ± 1 even if the initial remainder is $\pm\pi$.

If $N(\epsilon)=1$ then ϵ is a G-unit, so $\epsilon^3=\pm 1$

Since $\varphi + \psi = \chi$ and $\varphi = \pi\varphi'$, $\psi = \pi\psi'$, $\chi = \pi\chi'$

$\pi(\varphi' + \psi') = \pi(\chi') \Rightarrow \pi(\varphi' + \psi' - \chi') = 0$

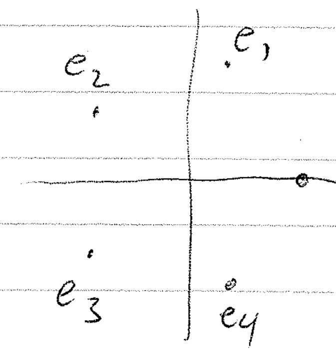
$\Rightarrow \varphi' + \psi' - \chi'$ in $\mathbb{C}$, ~~hence~~ (by dividing by π),

hence this is true as G-numbers,

and $\varphi' + \psi' = \chi'$.

By the above equation, if φ' and χ' have a common divisor δ, then δ also divides ψ'. But by the top of p. 99, φ', ψ' cannot have a common divisor.

② Let $e_1 = e^{\frac{2\pi i}{5}}$.



Then $e_k = (e_1)^k$, $k = 2, 3, 4, 5$

We have $(e_2+e_3)(e_1+e_4)$

$$= (e_1^2+e_1^3)(e_1+e_1^4)$$
$$= e_1^3+e_1^6+e_1^4+e_1^7 = e_1^3+e_1+e_1^4+e_1^2$$
$$= e_1+e_2+e_3+e_4 = -1 \text{ since}$$

e_1, e_2, e_3, e_4 are the roots of $x^4+x^3+x^2+x+1$.

Also $(e_2+e_3)+(e_1+e_4) = -1$.

So if we let $r = e_2+e_3$, $s = e_1+e_4$,

r and s are the roots of

$$x^2+x-1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}.$$

From the diagram, $r = e_2+e_3 < 0$, $s = e_1+e_4 > 0$,

so $r = \frac{-1-\sqrt{5}}{2}$, $s = \frac{-1+\sqrt{5}}{2}$

So $e_1+e_4 = \frac{-1+\sqrt{5}}{2}$

$e_1 \cdot e_4 = 1$ so

e_1, e_4 are the roots of

$$x^2 + \left(\frac{1-\sqrt{5}}{2}\right)x + 1 = 0$$

$$\Rightarrow x = \frac{\frac{-1+\sqrt{5}}{2} \pm \sqrt{\left(\frac{1-\sqrt{5}}{2}\right)^2 - 4}}{2} = \frac{-1+\sqrt{5}}{4} \pm \frac{\sqrt{\frac{3-\sqrt{5}}{2} - 4}}{2}$$

$= \frac{-1+\sqrt{5}}{4} \pm \frac{\sqrt{\frac{5+\sqrt{5}}{2}}}{2} i$. From the picture, $e_1 = \frac{-1+\sqrt{5}}{4} + \frac{\sqrt{\frac{5+\sqrt{5}}{2}}}{2} i$

③ We seek integers x, y such that $x\left(\frac{1}{3}\right)+y\left(\frac{1}{17}\right)=\frac{1}{51}$
ie. $17x+3y=1$. By use of the Euclidean algorithm, or guessing, $(x,y)=(-1,6)$ will do!

So we should construct first a point $6/17$ of the way around the circle, then backtrack $1/3$ of the way around, which gives us a point $6/17 - 1/3 = 1/51$ of the way around.