Math 510 HW #1 solutions

① $\sum_{i=1}^{n} i^6 = \frac{(n+x)^7 - x^7}{7}$

The text gives us $x \equiv 1/2,\ x^2 \equiv 1/6,\ x^3 \equiv 0,\ x^4 \equiv -1/30$

Next,

$$x^6-(x-1)^6 \equiv 0 \Rightarrow x^6-(x^6-6x^5+15x^4-20(x^3)+15x^2-6x+1) \equiv 0$$

$$\Rightarrow 6x^5 - 15(-1/30) + 20(0) - 15(1/6) + 6(1/2) - 1 \equiv 0$$

$$\Rightarrow 6x^5 + 1/2 - 5/2 + 3 - 1 \equiv 0$$

$$\Rightarrow 6x^5 \equiv 0 \Rightarrow x^5 \equiv 0$$

So $$\sum_{i=1}^{n} i^6 = \frac{n^7 + 7n^6x + 21n^5x^2 + 35n^4x^3 + 35n^3x^4 + 21n^2x^5 + 7nx^6}{7}$$

$$= \frac{n^7 + \frac{7}{2}n^6 + \frac{21}{6}n^5 + 0 - \frac{35}{30}n^3 + 0 + 7nx^6}{7}$$

~~(Forgot to get x^6;~~

$$0 \equiv x^7-(x-1)^7 \equiv x^7-(x^7-7x^6+21x^5-35x^4+35x^3-21x^2+7x-1)$$

$$\equiv 7x^6-21x^5+35x^4-35x^3+21x^2-7x+1$$

$$\equiv 7x^6-21(0)+35(-1/30)-35(0)+21(1/6)-7(1/2)+1$$

$$\equiv 7x^6 - 7/6 + 21/6 - 21/6 + 1$$

$$\Rightarrow x^6 \equiv 1/42$$

$$\frac{n^7 + \frac{7}{2}n^6 + \frac{7}{2}n^5 - \frac{7}{6}n^3 + \frac{1}{6}n}{7}$$

$$= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

M 510 HW #1

(2) Say $a \equiv b$ in $\mathbb{R}[x]/G$.

By definition, this means $a-b \in G$, ie.

$$a-b = c_1(x^2-(x-1)^2) + c_2(x^3-(x-1)^3) + c_3(x^4-(x-1)^4) \quad \text{in } \mathbb{R}[x]$$
$$= c_1(2x-1) + c_2(3x^2-3x+1) + c_3(4x^3-6x^2+4x-1)$$
$$= (-c_1+c_2-c_3) + x(2c_1-3c_2+4c_3) + x^2(3c_2-6c_3) + x^3(4c_3)$$

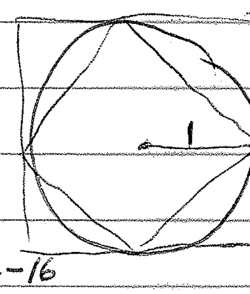
Thus
$$2c_1-3c_2+4c_3=0 \qquad 2c_1=0 \Rightarrow c_1=0$$
$$3c_2-6c_3=0 \qquad 3c_2=0 \Rightarrow c_2=0$$
$$4c_3=0 \Rightarrow c_3=0$$

So $a-b=0$ in $\mathbb{R}[x]$

ie $a=b$ in $\mathbb{R}$. ☒

(3) Fix any schoolgirl. If, over M days she ~~walks~~ walks with 2 other girls each day, she walks with a total of $2M$ other girls. So there are $2M+1$ girls total, which is odd, a contradiction.

(4) Starting with inscribed, circumscribed squares (see p. 186): $a_0=8$, $b_0=4\sqrt{2}$



Then we want b_2.

$$a_1 = \frac{2a_0b_0}{a_0+b_0} = \frac{2\cdot 8\cdot 4\sqrt{2}}{8+4\sqrt{2}} = \frac{16\sqrt{2}}{2+\sqrt{2}}\frac{(2-\sqrt{2})}{(2-\sqrt{2})} = 8\sqrt{2}(2-\sqrt{2}) = 16\sqrt{2}-16$$

$$b_1 = \sqrt{b_0a_1} = \sqrt{4\sqrt{2}(16\sqrt{2}-16)} = 8\sqrt{\sqrt{2}(\sqrt{2}-1)} = 8\sqrt{2-\sqrt{2}}$$

$$a_2 = \frac{2a_1b_1}{a_1+b_1} = \frac{2(16\sqrt{2}-16)(8\sqrt{2-\sqrt{2}})}{16\sqrt{2}-16+8\sqrt{2-\sqrt{2}}} = \frac{32(\sqrt{2}-1)\sqrt{2-\sqrt{2}}}{2\sqrt{2}-2+\sqrt{2-\sqrt{2}}}$$

$$b_2 = \sqrt{b_1a_2} = \sqrt{8\sqrt{2-\sqrt{2}}\,\frac{32(\sqrt{2}-1)\sqrt{2-\sqrt{2}}}{2\sqrt{2}-2+\sqrt{2-\sqrt{2}}}} = \text{desired perimeter}$$

⑤ $D_n - 2n D_{n-1} = (-1)^n$

Divide by $2^n n!$:

$$\frac{D_n}{2^n n!} - \frac{2n D_{n-1}}{2^n n!} = \frac{(-1)^n}{2^n n!}$$

$$\frac{D_n}{2^n n!} - \frac{D_{n-1}}{2^{n-1}(n-1)!} = \frac{(-1)^n}{2^n n!}$$

$$\sum_{k=1}^{n} \underbrace{\frac{D_k}{2^k k!} - \frac{D_{k-1}}{2^{k-1}(k-1)!}}_{\text{telescopes!}} = \frac{(-1)^k}{2^k k!}$$

$$\frac{D_n}{2^n n!} - \frac{D_0}{1} = \sum_{k=1}^{n} \frac{(-1)^k}{2^k k!}$$

$D_0 = 1$ (assumption)

$$\frac{D_n}{2^n n!} = 1 + \sum_{k=1}^{n} \frac{(-1)^k}{2^k k!} = \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!}$$

$$D_n = 2^n n! \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!}$$