Math 430

Exam 1 solutions

(1)

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & -3 & 1 & -2 \\ & & -2 & 4 & -2 & 2 \\ \hline & 1 & -2 & 1 & -1 & 0 \end{array}$$

$$(x^4-3x^2+x-2) = (x+2)(x^3-2x^2+x-1)$$

So $\displaystyle\lim_{x\to -2} \frac{(x^4-3x^2+x-2)}{(x+2)} = \lim_{x\to -2}(x^3-2x^2+x-1)$

$= -8-8-2-1 = \boxed{-19}$

(2) $\displaystyle\lim_{x\to 0} \frac{1+\binom{n}{1}mx+\binom{n}{2}(mx)^2 + \text{higher powers of } x - \left(1+\binom{m}{1}nx+\binom{m}{2}(nx)^2+\cdots\right)}{x^2}$

$= \displaystyle\lim_{x\to 0} \frac{1+nmx+\frac{n(n-1)}{2}m^2x^2+\text{higher powers} - \left(1+mnx+\frac{m(m-1)}{2}n^2x^2+\cdots\right)}{x^2}$

$= \displaystyle\lim_{x\to 0}\left(\frac{n(n-1)}{2}m^2-\frac{m(m-1)}{2}n^2\right) + \text{terms with } x$

$= \binom{n}{2}m^2 - \binom{m}{2}n^2$

(3) b is an upper bound for A if $a \le b \ \forall a \in A$.

b is a least upper bound for A if b is an upper bound for A and $b \le c \ \forall$ c that is an upper bound for A

$\mathbb{R}$ is <u>complete</u>: means that every set $A \subseteq \mathbb{R}$ which is bounded above has a least upper bound.

(4)(a) $|x+y| \le |x|+|y|$

$|x-y| \ge ||x|-|y||$

(b) $\forall M>0 \ \exists \ \delta>0 \ni$ if $0<|x-a|<\delta$ then $f(x)>M$.

(5) Know | Want

$|x-3|<\delta$

$\delta=1: 2<x<4$

$4< x+2<6$

$\frac{1}{6}<\frac{1}{x+2}<\frac{1}{4}$

$\left|\frac{2(x-3)}{5(x+2)}\right| \le \frac{2}{5}\cdot\frac{\delta}{4}=\frac{\delta}{10}$

$\delta = 10\epsilon$

$\left|\frac{x}{x+2}-\frac{3}{5}\right| = \left|\frac{5x-3x-6}{5(x+2)}\right| = \left|\frac{2(x-3)}{5(x+2)}\right|$

Pf Fix $\epsilon>0$. Let $\delta=\min\{1, 10\epsilon\}$ then if $0<|x-3|<\delta$,

$0<|x-3|<1 \Rightarrow 2<x<4 \Rightarrow 4<x+2<6 \Rightarrow \frac{1}{x+2}<\frac{1}{4}$

Also $|x-3|<10\epsilon$

Then $\left|\frac{x}{x+2}-\frac{3}{5}\right| = \left|\frac{5x-3x-6}{5(x+2)}\right| = \left|\frac{2(x-3)}{5(x+2)}\right| \le \frac{2\cdot 10\epsilon}{5\cdot 4} = \epsilon$

Since ϵ was arbitrary, $\lim_{x\to 3}\frac{x}{x+2}=\frac{3}{5}$ by def of limit.

(6) Since $-1\le \sin\left(\frac{1}{x^2}\right)\le 1$, $-x^4 \le x^4\sin\left(\frac{1}{x^2}\right)\le x^4$

and since $\lim_{x\to 0} x^4 = \lim_{x\to 0} -x^4 = 0$,

$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x^2}\right) = 0$ by the squeeze theorem

(7) Recall that $0 < \cos x < \frac{\sin x}{x} < 1$ for $0 < |x| < \pi/2$.

Then $\cos x - 1 < \frac{\sin(x)}{x} - 1 < 0$

$\Rightarrow \quad 0 < 1 - \frac{\sin x}{x} < 1 - \cos x \Rightarrow 0 < \left|1 - \frac{\sin x}{x}\right| < |1 - \cos x|$

But $|1 - \cos x| = |2 \sin^2 \frac{x}{2}| \leq 2 |\sin \frac{x}{2}| |\sin \frac{x}{2}| \leq 2 \cdot 1 \cdot |\frac{x}{2}| = |x|$

Now fix $\epsilon > 0$ and let $\delta = \min\{\epsilon, \frac{\pi}{2}\}$.

if $0 < |x| < \delta$ then $\underbrace{\left|1 - \frac{\sin x}{x}\right| < |x|}_{\text{since } 0<|x|<\pi/2} < \epsilon$.

Since ϵ was arbitrary, $\lim_{x \to 0} \frac{\sin x}{x} = 1$ by def of limit.

(8) See class notes.

(9) Know | Want

$0 < |x-3| < \delta$

if $\delta < 1$, $\boxed{2 < x} < 4$ | $\frac{x}{(x-3)^2} > \frac{2}{\delta^2}$. Let $\delta = \sqrt{\frac{2}{M}}$

$\frac{1}{(x-3)^2} > \frac{1}{\delta^2}$ | $\frac{x}{(x-3)^2} > M$

Pf

Fix $M > 0$. Let $\delta = \min\{1, \sqrt{\frac{2}{M}}\}$. Then if $0 < |x-3| < \delta$,

$0 < |x-3| < 1 \Rightarrow 2 < x < 4$ and $\frac{1}{(x-3)^2} > \frac{1}{\delta^2} > \frac{1}{\left(\frac{2}{M}\right)} = \frac{M}{2}$

so $x \cdot \frac{1}{(x-3)^2} > 2 \cdot \frac{M}{2} = M$.

Since M was arbitrary, $\lim_{x \to 3} \frac{x}{(x-3)^2} = +\infty$ by def of limit.

(10) See text/class notes.