Problem Set 26, p. 193

1. 1112. **3.** a) 1; b) 2; c) ∞; d) 1; e) 0; f) $\frac{2}{9}$; g) No limit; h) 0; i) No limit. **5.** a) $x_n = \frac{1}{n^2}$, $y_n = \frac{1}{n}$; b) $x_n = \frac{1}{n}$, $y_n = \frac{1}{n}$; c) $x_n = \frac{1}{n}$, $y_n = \frac{1}{n^2}$; d) $x_n = \frac{1}{n}$, $y_n = \frac{(-1)^n}{n}$. **7.** a) 1 if $|a| > 1$, 0 if $|a| < 1$, $\frac{1}{2}$ if $a = 1$; b) 0 if $|a| > 1$ or $|a| < 1$, $\frac{1}{2}$ if $a = 1$, no limit if $a = -1$. **9.** a) $x_n = (-\frac{1}{2})^n$; b) $x_n = \frac{1}{n}$; c) $x_n = \frac{n-1}{n}$; d) $x_n = (-1)^n \frac{n-1}{n}$. **11.** a) $1/e$; b) e. **13.** If $x_n \to c$ as $n \to \infty$, then every neighborhood of c contains all but a finite number of terms of $\{x_n\}$ and hence infinitely many terms of $\{x_n\}$, i.e., c is an accumulation point of $\{x_n\}$. The sequence $\{(-1)^n\}$ has two accumulation points (-1 and $+1$) but no limit. **15.** The sequence $\{x_n\}$ is obviously increasing. The recursion formula $x_n = \sqrt{a + x_{n-1}}$ implies $x_n^2 = a + x_{n-1}$ and hence

$$x_n = \frac{a}{x_n} + \frac{x_{n-1}}{x_n} < \frac{a}{x_n} + 1$$

since $x_{n-1} < x_n$. But $x_n > \sqrt{a}$ and hence $x_n < \sqrt{a} + 1$ for all n. Therefore $\{x_n\}$ is bounded and increasing. It follows from Theorem 4.17 that $\{x_n\}$ is convergent, with limit c. To find c, note that

$$c^2 = \lim_{n\to\infty} x_n^2 = a + \lim_{n\to\infty} x_n = a + c,$$

and hence

$$c^2 = a + c. \tag{1}$$

Therefore

$$c = \frac{1 + \sqrt{1 + 4a}}{2},$$

since the negative solution of (1) is excluded (why?). **17.** The "rational underestimates" of $\sqrt{2}$ (cf. p. 64), i.e., 1.4, 1.41, 1.414, 1.4142, . . .

Problem Set 27, p. 202

1. f is discontinuous at $x = 1$ but continuous elsewhere; g is continuous at every point of [0, 2]. **3.** The function

$$f(x) = \begin{cases} x \text{ if } x \text{ is rational,} \\ -x \text{ if } x \text{ is irrational} \end{cases}$$

is continuous at $x = 0$ but discontinuous everywhere else. **5.** If $f(x_0) \neq 0$, then by Theorem 4.3 there exists a $\delta < 0$ such that $f(x)$ has the same sign as $f(x_0)$ if $0 < |x - x_0| < \delta$ and hence obviously if $|x - x_0| < \delta$. **7.** a) $-\frac{3}{2}$; b) $\frac{1}{2}$; c) No choice; d) 0; e) 2; f) No choice. **9.** $\rho(x)$ is given by

$$\rho(x) = \begin{cases} -x & \text{if } x < 0, \\ 0 & \text{if } 0 \le x \le 1, \\ x - 1 & \text{if } 1 < x \le \frac{3}{2}, \\ 2 - x & \text{if } \frac{3}{2} < x < 2, \\ 0 & \text{if } 2 \le x \le 3, \\ x - 3 & \text{if } x > 3, \end{cases}$$

and is continuous everywhere. **11.** If f and g are both discontinuous at x_0, fg may still be continuous. For example, the function $f(x)$ given in the answer to Prob. 3 is discontinuous at any point $x_0 \neq 0$, but its square is continuous at x_0. Of course, the product fg of two discontinuous functions may be discontinuous (e.g., $x_0 = 0$, $f(x) = g(x) = 1/x$). If f is continuous and g discontinuous at x_0, then fg may be continuous (e.g., $x_0 = 0$, $f(x) = x$, $g(x) = \sin(1/x)$) or discontinuous (e.g., $x_0 = 0$, $f(x) = x$, $g(x) = 1/x^2$). **13.** $A = -1$, $B = 1$. **15.** a) $f \circ g$ has no discontinuities; $g \circ f$ is discontinuous at 0; b) $f \circ g$ has discontinuities at -1, 0 and 1; $g \circ f$ has no discontinuities; c) $f \circ g$ and $g \circ f$ have no discontinuities.

17. a) $f(x) = \begin{cases} \frac{1}{x} \text{ if } x < 0, \\ x \text{ if } x \ge 0; \end{cases}$ b) $f(x) = \begin{cases} x & \text{if } x \le 0, \\ \sin \frac{1}{x} & \text{if } x > 0. \end{cases}$

19. a) $f(x)$ has a discontinuity at $x = 1$, since

$$f(x) = \begin{cases} 1 \text{ if } 0 \le x < 1, \\ \frac{1}{2} \text{ if } x = 1, \\ 0 \text{ if } x > 1; \end{cases}$$

b) $f(x)$ has a discontinuity at every point $x = n\pi$ ($n = 0, \pm 1, \pm 2, \ldots$), since

$$f(x) = \begin{cases} 0 \text{ if } x \neq n\pi, \\ 1 \text{ if } x = n\pi. \end{cases}$$

21. $\Delta x = -0.009$, $\Delta y = 990{,}000$. **23.** If $\Delta f = f(x_0 + \Delta x) - f(x_0)$, then

$$\lim_{\Delta x \to 0+} \Delta f = 0$$

means that $f(x)$ is continuous from the right at x_0, while

$$\lim_{\Delta x \to 0-} \Delta f = 0$$

means that $f(x)$ is continuous from the left at x_0.