Sec. 26

3a $\lim_{n\to\infty} \frac{3^n - 1}{3^n} = \lim_{n\to\infty} 1 - \frac{1}{3^n} = 1 - 0 = 1$

3b $\lim_{n\to\infty} \sum_{i=0}^{n} \left(\frac{1}{2}\right)^i = \lim_{n\to\infty} \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \frac{1-0}{1-\frac{1}{2}} = 2$

h) Since $-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n} \quad \forall n \in \mathbb{N}$

and $\lim_{n\to\infty} \pm\frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{\cos n}{n} = 0$ by squeeze.

8 $a^n = (1+(a-1))^n = 1 + \binom{n}{1}(a-1) + \binom{n}{2}(a-1)^2 + \cdots \binom{n}{n}(a-1)^n$

$\ge 0 + 0 + \binom{n}{2}(a-1)^2 + 0 + 0 + 0$

$= \frac{n(n-1)}{2}(a-1)^2$

Note that $0 \le \frac{n}{a^n} \le \frac{n}{\frac{n(n-1)}{2}(a-1)^2} = \frac{2}{(n-1)(a-1)^2}$

Since $\lim_{n\to\infty} \frac{2}{(n-1)(a-1)^2} = 0$, $\lim_{n\to\infty} \frac{n}{a^n} = 0$

by squeeze.

18 Fix $\epsilon > 0$. Choose $M \in \mathbb{R}$, $M > 0 \ni$

$x > M \Rightarrow |f(x) - c| < \epsilon$.

Let $N = [M] + 1$. Then if $n > N$, $n > M$,

so $|f(n) - c| < \epsilon$.

Since ϵ was arbitrary, $\lim_{n\to\infty} f(n) = c$

Sec 27

18 It was shown in class that $\forall c \in \mathbb{R}$, $\lim_{x \to c} |x| = |c|$. Thus $f(x) = |x|$ is continuous every where.

Since f and g are continuous everywhere, so are $\frac{1}{2}(f+g)$ and $\frac{1}{2}(f-g)$ by corollaries 1 and 2.

Thus $\frac{1}{2}|f-g|$ is also, by Theorem 4.20 (composing $\frac{1}{2}(f-g)$ with the absolute value)

Thus $\max\{f, g\} = \frac{1}{2}|f-g| + \frac{1}{2}(f+g)$ is continuous everywhere by Corollary 1.

Then $\min\{f,g\} = -\max\{-f,-g\}$. If f and g are continuous everywhere, so are $-f, -g$ (cor. 2), so $\max\{-f,-g\}$ is (by the work above), so $-\max\{-f,-g\}$ is also, by cor. 2.

Note. If $f(x) \geq g(x)$,

$$\frac{1}{2}|f(x)-g(x)| + \frac{1}{2}(f(x)+g(x)) = \frac{1}{2}(f(x)-g(x)) + \frac{1}{2}(f(x)+g(x)) = f(x)$$

If $f(x) \leq g(x)$,

$$\frac{1}{2}|f(x)-g(x)| + \frac{1}{2}(f(x)+g(x)) = \frac{1}{2}(g(x)-f(x)) + \frac{1}{2}(f(x)+g(x)) = g(x)$$

Either way, $\frac{1}{2}|f(x)-g(x)| + \frac{1}{2}(f(x)+g(x)) = \max\{f(x), g(x)\}$

Sec 27

(20) Let $r = m/n$ be rational. Then $f(r) = 1/n > 0$.

Let $\epsilon = 1/2n$. Then given any $\delta > 0$ $\exists$ $t_\delta \ni$

$0 < |t_\delta - r| < \delta$ but $|f(t_\delta) - f(r)| = \frac{1}{n} > \epsilon$.

So it is false that $\forall \epsilon > 0$ $\exists$ $\delta > 0 \ni$

$0 < |x - r| < \delta \Rightarrow |f(x) - f(r)| < \epsilon$; it fails for $\epsilon = \frac{1}{2n}$.

Thus f is discontinuous at $x = r$.

Let r be irrational. Then $f(r) = 0$. We show $\lim_{x \to r} f(x) = 0$. Fix $\epsilon > 0$. Choose integer $N \ni N > \frac{1}{\epsilon}$. Let δ be the distance to the nearest rational number with denominator $\leq N$.

Then if $0 < |x - r| < \delta$, $f(x)$ either $= 0$ (if x is irrational) or $f(x) < \frac{1}{N} < \epsilon$.

Either way, $0 \leq f(x) < \epsilon$, so $|f(x) - 0| < \epsilon$ for $0 < |x - r| < \delta$.

Since ϵ is arbitrary, $\lim_{x \to r} f(x) = 0$

Sec 29

(2) If f, g are diff'ble then $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

and $g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} \neq 0$

so $$\frac{f'(0)}{g'(0)} = \frac{\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}}{\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}} = \lim_{x\to 0}\frac{\frac{f(x)-f(0)}{x-0}}{\frac{g(x)-g(0)}{x-0}} = \lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)} = \lim_{x\to 0}\frac{f(x)}{g(x)}$$

(8b) $$y' = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h\,(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$y'' = \lim_{h\to 0}\frac{\frac{1}{2\sqrt{x+h}} - \frac{1}{2\sqrt{x}}}{h} = -\frac{1}{2}\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h\sqrt{x+h}\sqrt{x}}$$

$$= -\frac{1}{2}\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{1}{\sqrt{x+h}\sqrt{x}}\right) = -\frac{1}{2}\left(\frac{1}{2\sqrt{x}}\right)\left(\frac{1}{\sqrt{x}\sqrt{x}}\right) = -\frac{1}{4}x^{-3/2}$$

Sec. 31

(6b) $$y' = \frac{(1-x+x^2)(1-2x)-(1+x-x^2)(-1+2x)}{(1-x+x^2)^2} = \frac{(1-2x)(1-x+x^2+1+x-x^2)}{(1-x+x^2)^2}$$

$$= \frac{2(1-2x)}{(1-x+x^2)^2}$$

(6d) $$y' = \frac{(1-x)^2[-2x(3-x^3)+(2-x^2)(-3x^2)] - (2-x^2)(3-x^3)2(1-x)(-1)}{(1-x)^4}$$

$$= \frac{(1-x)\left((1-x)[-6x+2x^4-6x^2+3x^4] + 2x^5-4x^3-6x^2+12\right.}{(1-x)^4}$$

$$= \frac{(1-x)(5x^4-6x^2-6x)+2x^5-4x^3-6x^2+12}{(1-x)^3}$$