Sparse quasi-random graphs

(Fan Chung and Ron Graham)

What is special about a random graph?

Some notation:

$G = (V,E)$ – graph, $v(G)$ = # of vertices, $e(G)$ = # of edges of G

$nd(x) = \{y \in V: \{x,y\} \in E\}$, $deg(x) = |\, nd(x)\, |$

$G(n)$ – graph with n vertices

Random graphs

$G_{1/2}(n)$: $V = \{1,2,3, \dots, n\}$

Each pair $\{i,j\}$ is chosen to be an edge

with independent probability 1/2

This defines a probability distribution of the set of all $2^{\binom{n}{2}}$ (labelled) graphs on n vertices, namely each such graph has probability $2^{-\binom{n}{2}}$.

Typical result for a random graph

$$\boxed{e(G_{1/2}(n)) = (1+o(1))\, n^2/4}$$

In other words,

For all $\varepsilon > 0$, if $n > n_0(\varepsilon)$ then

$$Pr(|\, e(G_{1/2}(n) - n^2/4\,| > \varepsilon n^2) < \varepsilon$$

Sometimes we say:

Almost all graphs on n vertices have $(1+o(1))\, n^2/4$ edges.

The Probabilistic Method (the Erdös method)

1. Assign suitable probability measure to space of objects;

2. Show that the set of "good" objects has positive probability;

3. Conclude that "good" objects exist.

Remaining problem: Find one!

Ramsey number $r(n)$ = largest number r such that every graph $G(n)$ has either a complete subgraph of size r, or an independent set of size r.

Theorem (Erdös – 1947):

$$\tfrac{1}{2}\log_2 n < r(n) < 2\log_2 n$$

How ? The probabilistic method !

Almost all graphs $G = G(n)$ have no complete subgraph and no independent set of size $2\log_2 n$

However, no one can (yet) construct such graphs.

The best construction only gives

$$r(n) < \exp(c\sqrt{\log n})$$

Ramsey number $r(n)$ = largest number r such that every graph $G(n)$ has either a complete subgraph of size r, or an independent set of size r.

Theorem (Erdös – 1947):

$$\tfrac{1}{2} \log_2 n < r(n) < 2 \log_2 n$$

How ? The probabilistic method

Almost all graphs $G = G(n)$ have no complete subgraph and no independent set of size $2 \log_2 n$

Challenge ($1000)

Show that $\lim_{n\to\infty} (r(n)/\log_2 n)$ exists.

Property P : $\sum_v |\deg(v) - n/2| = o(n^2)$

Property P′ : All but o(n) vertices v have deg(v) = (1 + o(1)) n/2

To say for a family G(n) that P $\Rightarrow$ P′ means:

For all $\varepsilon' > 0$ there exists $\varepsilon > 0$ such that if $n > n_0(\varepsilon)$ and

$\sum_v |\deg(v) - n/2| < \varepsilon n^2$ then all but $\varepsilon' n$ vertices v have

$$\sum_v |\deg(v) - n/2| < \varepsilon' n^2$$

In fact, P $\Leftrightarrow$ P′ by Cauchy-Schwarz.

The philosophy of quasi-randomness:

There exist a large class of graph properties P(1), P(2), P(3),… such that:

(1) Each P(i) is possessed by almost all $G_{1/2}(n)$;

(2) All the P(i) are equivalent, i.e., if $G(n)$ satisfies some P(i)

then it must satisfy all of the other P(j);

(3) It is often easy to show that particular graph families satisfy

at least one (and therefore all) of properties P(i).

We call graph families satisfying these properties quasi-random.

We call properties in this equivalence class quasi-random graph properties.

G, H - graphs

H is an induced subgraph of G (written $H \prec G$) if there exists

$\lambda : V(H) \to V(G)$ so that $\{x,y\} \in E(H)$ if and only if $\{\lambda(x),\lambda(y)\} \in E(G)$

$\#\{H \prec G\}$ denotes the number of such λ.

Note that if $v(H) = s$ then $\#\{H \prec G_{1/2}(n)\} = (1+o(1))n^s 2^{-\binom{s}{2}}$

In other words, all s-vertex (labelled) graphs H occur asymptotically equally often as induced subgraphs of $G_{1/2}(n)$

We say that $\{G(n)\}$ has the **induced s-subgraph** property if

for every fixed graph $H(s)$ on s vertices,

$$\#\{H(s) \prec G(n)\} = (1 + o(1))n^s 2^{-\binom{s}{2}}$$

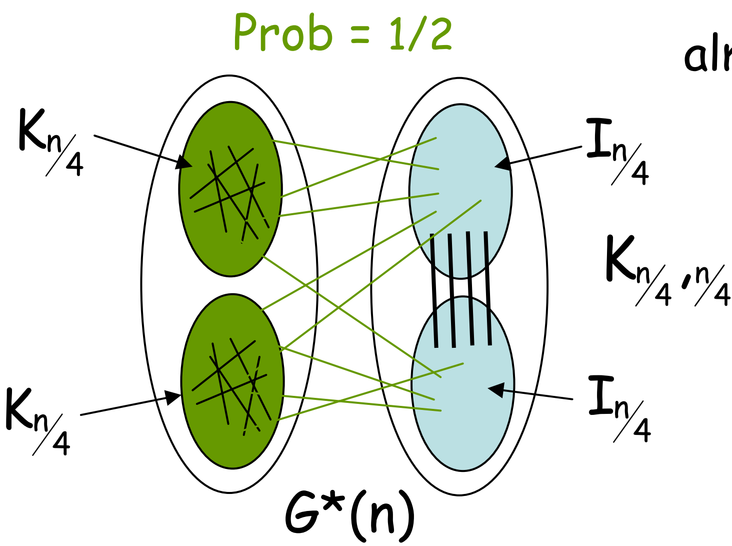
$\ldots\ldots \Rightarrow$ induced (s+1)-subgraph $\Rightarrow$ induced s-subgraph $\Rightarrow \ldots\ldots$

$\ldots\ldots \Rightarrow$ induced 4-subgraph $\underset{\not\Leftarrow}{\Rightarrow}$ induced 3-subgraph $\underset{\not\Leftarrow}{\Rightarrow}$ induced 2-subgraph

$\Downarrow \; \not\Uparrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\Updownarrow$

almost regular $\underset{\not\Leftarrow}{\Rightarrow}$ $e(G(n)) = (1+o(1))\, n^2/4$

$K_{n/2,\, n/2}$



G*(n) has the induced 3-subgraph property but it does not have the induced 4-subgraph property

For example, $\#\{K_4 \prec G^*(n)\} = (1+o(1))\frac{11}{512}n^4$ and $\frac{11}{512} > \frac{1}{2^6}$

Induced s-subgraph property $(s \geq 4)$

For every fixed graph H(s) on s vertices,

$$\#\{H(s) \prec G(n)\} = (1 + o(1))n^s 2^{-\binom{s}{2}}$$

4-cycle property:

$$\#\{C_4 \subset G(n)\} \le (1+o(1))\frac{n^4}{16} \qquad \text{and} \qquad e(G(n)) \ge (1+o(1))\frac{n^2}{4}$$

Note that C_4 is just a subgraph of $G(n)$ (not necessarily induced).

Uniform edge density property:

For every subset $S \subseteq V$, $\quad e(S) = (1+o(1))\frac{|S|^2}{4} + o(n^2)$

(where $e(S)$ denotes the # of edges spanned by S).

$A(G) = (a_{u,v})_{u,v \in V}$ - adjacency matrix of $G = G(n)$ defined by:

$$a_{u,v} = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ 0 & \text{otherwise} \end{cases}$$

$A(G)$ is real and symmetric, so has real eigenvalues

$$\lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq \ldots\ldots \geq |\lambda_n|$$

Separated eigenvalues property

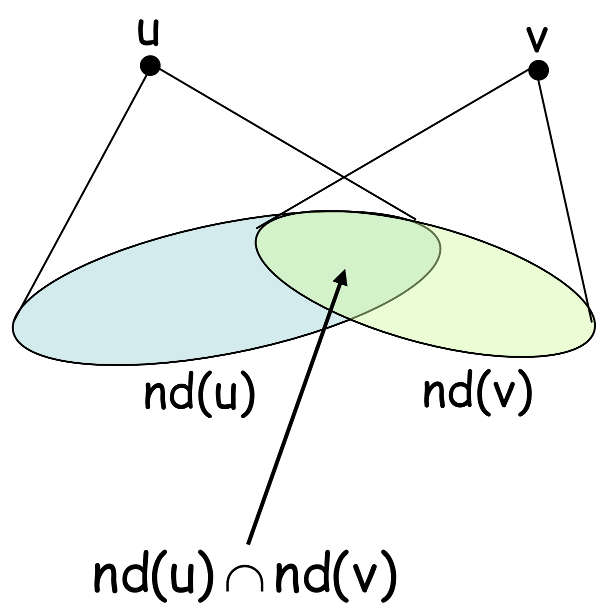
$$\lambda_1 = (1 + o(1))\, n/2,\ \lambda_2 = o(n), \text{ and } \ e(G(n)) \geq (1 + o(1))\frac{n^2}{4}$$

Note: For $G_{1/2}(n)$,

$$\lambda_1 = (1 + o(1))\, n/2, \quad \lambda_2 = O(\sqrt{n})$$

Neighborhood property

$$\sum_{u,v} \left| \, |nd(u) \cap nd(v)| - n/4 \, \right| = o(n^3)$$



Theorem (Chung, Graham and Wilson)

All of the properties 4-cycle, separated eigenvalues, neighborhood, uniform density and induced s-subgraph, $s \geq 4$, (and many, many others) are quasi-random graph properties.

In particular, this implies that if a graph (family) has the expected number of edges and 4-cycles, then for any fixed s, it contains asymptotically the expected number of induced subgraphs isomorphic to any fixed s-vertex graph H(s).

Sample implication

4-cycle $\Rightarrow$ separated eigenvalues

By hypothesis $\#\{C_4 \subseteq G(n)\} \le (1+o(1))\frac{n^4}{16}$, $e(G(n)) \ge (1+o(1))\frac{n^2}{4}$

Denote the eigenvalues of $A(G(n))$ by $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ where

$$\lambda_1 \ge |\lambda_2| \ge |\lambda_3| \ge \ldots\ldots \ge |\lambda_n|$$

For $\bar{v} = (1,1,\ldots,1)^{tr}$ we have

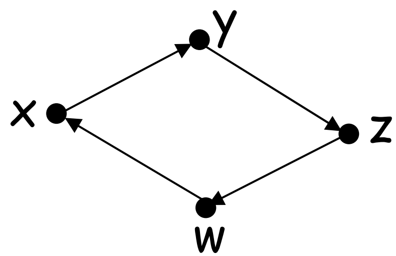
$$\lambda_1 \ge \frac{\langle A\bar{v}, \bar{v}\rangle}{\langle \bar{v}, \bar{v}\rangle} = \frac{1}{n}\sum_v \deg(v) \ge (1+o(1))\, n/2$$

Consider tr(A^4), the trace of A^4.

On one hand, $\quad \mathrm{tr}(A^4) = \sum_{i=1}^{n} \lambda^4 \geq \lambda_1^4 \geq (1+o(1))n^4/16$

One the other hand, which terms contribute to tr(A^4)?

$$A = \begin{bmatrix} & \vdots & \\ \cdots & a_{x,x} & \cdots \\ & \vdots & \end{bmatrix}$$



$\therefore\ \mathrm{tr}(A^4) = \#\{C_4 \subseteq G(n)\} \leq (1+o(1))\,n^4/16$. Thus,

$$\sum_{i=1}^{n} \lambda^4 = \mathrm{tr}(A^4) = (1+o(1))n^4/16,\ \lambda_1 > 0, \text{ and so } \lambda_1 = (1+o(1))n/2$$

and $\displaystyle\sum_{i=2}^{n} |\lambda_i|^4 = o(n^4)$. Since λ_2 is real then $\lambda_2 = o(n)$.

Therefore, the **separated eigenvalues** property holds. ■

Examples of quasi-random graph families

(1) Quadratic residue graphs QR(p)

$V = \mathbb{Z}/\mathbb{Z}_p$, $p \equiv 1 \pmod 4$ prime

$\{i,j\} \in E$ if and only if $i - j$ is a quadratic residue modulo p

(2) $\cap(n)$ - even intersection graphs

V = all n-sets of $\{1, 2, \ldots\ldots, 2n\}$

$\{X,Y\} \in E$ if and only $|X \cap Y|$ is even.

(3) For any fixed $\alpha > 0$, if $n > n_0(\alpha)$ and $G(n)$ is quasi-random,

then any family of induced subgraphs $H \prec G(n)$

with $v(H) > \alpha n$ is also quasi-random.

Tensor product graph families

$G = (V, E)$, $G' = (V', E')$ - graphs

$G \otimes G' = (V^*, E^*)$ where $V^* = V \times V'$, and $\{(u,u'), (v,v')\} \in E$ if and only if

either

$\{u,v\} \in E$ and $\{u',v'\} \notin E'$,

or

$\{u,v\} \notin E$ and $\{u',v'\} \in E'$,

Theorem

If G is not a complete bipartite graph, then the family

$$\{G^{\otimes n} := \overbrace{G \otimes G \otimes \ldots\ldots \otimes G}^{n \text{ terms}}, n \to \infty\}$$ is quasi-random.

What happens when the edge density is not $\frac{1}{2}$?

If $G(n)$ has $e(G(n)) \sim (1+o(1))\alpha n^2$ for some $\alpha > 0$,

then everything still goes through (except the $\otimes$- product construction).

However, when $e(G(n)) = o(n^2)$ then the situation becomes much more subtle.

We will focus on this situation for the rest of the talk, and particularly on the most difficult range in which $e(G(n)) = o(n^{3/2})$.

In this case, for example, $G(n)$ might not have any "real" 4-cycles

We assume that $\mathfrak{G}_p$ is a family of graphs $G = G(n) = (V, E)$ satisfying $e(G) = (1 + o(1))p\binom{n}{2}$, $n \to \infty$, where we always assume that $p = p(n)$ satisfies $pn \to \infty$ as $n \to \infty$.

We next list a collection of graph properties that will be used in what follows.

DISC : For all $X \subset V$,

$$| e(X) - p \binom{|X|}{2} | = o(pn^2)$$

(where e(X) denotes the # of edges spanned by X).

DISC(1) : For all $X, Y \subset V$,

$$| e(X,Y) - p|X||Y| | = o(pn^2)$$

(where e(X,Y) denotes the # of edges between X and Y).

For $X, Y \subset V$, let $e_t(X,Y)$ denote the number of length t walks $v_0, v_1, \ldots, v_t$

i.e., $v_0 \in X$, $v_t \in Y$, and $v_i \sim v_{i+1}$, for $1 \le i < t$.

Note: We do **not** require that all vertices or edges in a walk to be distinct.

DISC(t) : For all $X, Y \subset V$,

$$| e_t(X,Y) - p^t n^{t-1} |X||Y| | = o(p^t n^{t+1})$$

By the eigenvalues $\lambda_i = \lambda_i(G)$ of G, we mean the eigenvalues of the adjacency matrix of G arranged so that

$\lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$

EIG: The eigenvalues $\lambda_i = \lambda_i(G)$ satisfy:

$$\lambda_1 = (1 + o(1))pn,$$

$$\lambda_i = o(pn),\ i > 1.$$

A related condition is the following (where t is a positive integer).

EIG(t): The eigenvalues $\lambda_i = \lambda_i(G)$ satisfy:

$$\sum_i |\lambda_i|^t = (1 + o(1))(pn)^t$$

By a t-circuit C_t^* we mean a walk $v_0, v_1, \ldots, v_t$ of length t with $v_0 = v_t$.

CIRCUIT(t): The number of t-circuits C_t^* in G satisfies:

$$\#\{C_t^* \subset G\} = (1 + o(1))(pn)^t.$$

A cycle is a circuit in which all the vertices are required to be distinct, so that a t-cycle C_t has t distinct vertices.

An analogous property for t-cycles is the following:

CYCLE(t): The number of t-cycles C_t in G satisfies:

$$\#_{1-1}\{C_t \subset G\} = (1 + o(1))(pn)^t, \quad \text{where}$$

$$\#_{1-1}\{C_t \subset G\} = |\{\rho : V(C_t) \to V(G) \text{ such that } x \sim y \text{ in } C_t \Rightarrow \rho(x) \sim \rho(y) \text{ in } G, \text{ and } \rho \text{ is injective}\}|$$

We point out that the common value for **EIG**(2t), **CIRCUIT**(2t) and **CYCLE**(2t), namely $(1+o(1))(pn)^{2t}$, is what you expect for a p-random graph $G_p(n)$ when $p \gg n^{-1+1/t}$.

When p is much smaller, however, then this is not what happens in a p-random graph. For example, when $p = n^{-3/4}$, then $G_p(n)$ would be expected to have $(1+o(1))p^2n^3 = (1+o(1))n^{3/2}$ 4-circuits, which is a lot larger than $(1+o(1))(pn)^4 = (1+o(1))n$, the expected number of 4-cycles in $G_p(n)$.

Finally, we have the (very weak) condition:

AR: $$\sum_{v \in V} |\deg(v) - pn| = o(pn^2)$$

(i.e., G is "almost regular").

Theorem:

For any family $\mathfrak{G}_p$ with $pn \to \infty$ as $n \to \infty$, the following implications hold for all $t \geq 1$:

$$
\begin{array}{ccccccccc}
\mathrm{CIRCUIT}(2t) & \Leftrightarrow & \mathrm{EIG}(2t) & \Rightarrow & \mathrm{EIG} & \Rightarrow & \mathrm{DISC} & \Leftrightarrow & \mathrm{DISC}(1) \\
 & & & & \Downarrow & & \Downarrow & & \\
 & & & & \mathrm{DISC}(t) & & \mathrm{AR} & &
\end{array}
$$

Sample proof (sketch): $\boxed{\text{DISC} \Rightarrow \text{EIG}}$

Step 1 : DISC $\Rightarrow$ AR

Proof: Let $\bar{v} = (1, 1, \ldots\ldots, 1)^{tr}$. For the adjacency matrix A of G with eigenvalues $\lambda_1 \geq |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$, we have $\|A\bar{v}\| \leq \lambda_1 \|v\|$.

Thus,

$$\sum_x \deg(x)^2 \leq (1+o(1))p^2n^3.$$

On the other hand, $e(G) = \frac{1}{2}\sum_x \deg(x) \geq (1+o(1))p\binom{n}{2}$.

By Cauchy-Schwarz, we have

$$(1+o(1))p^2n^3 \geq \sum_x (\deg(x))^2 \geq \tfrac{1}{n}(\sum_x \deg(x))^2 = (1+o(1))p^2n^3.$$

which implies (by Cauchy-Schwarz again) that $\sum_x |\deg(x) - pn| = o(pn^2)$, as required.

<u>Step 2</u> : Let us denote by $\bar{e}_1$ the eigenvector associated with λ_1, normalized so that $\|\bar{e}_1\| = 1$. Then EIG implies that

$$(*) \qquad \|\bar{u} - \bar{e}_1\| = o(1) \quad \text{where} \quad \bar{u} = \tfrac{1}{\sqrt{n}}(1,1,\ldots,1)^{tr}$$

Proof: Suppose we write $\bar{u} = \sum_i a_i \bar{e}_i$ where the $\bar{e}_i$ are orthonormal eigenvectors of A. Thus, $A\bar{u} = \sum_i a_i \lambda_i \bar{e}_i$.

By Step 1, we have $A\bar{u} = pn\bar{u} + \bar{w}$ where $\|\bar{w}\| = o(pn)$. Consequently,

$$\bar{w} = \sum_i a_i \lambda_i \bar{e}_i - pn\bar{u} = \sum_i (\lambda_i - pn) a_i \bar{e}_i$$

and so

$$\sum_i (\lambda_i - pn)^2 a_i^2 = o(p^2 n^2).$$

Since $\bar{u} = a_1 \bar{e}_1$ with $\|\bar{w}_1\| = o(1)$ and $\|\bar{u}\| = \|\bar{e}_1\| = 1$ then $|a_1| = 1 + o(1)$.

This implies that $a_1 = 1 + o(1)$ and (*) is proved.

<u>Step 3</u>: For $S \subset V$, define $\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$

Then

$$2e(S) = \langle \chi_S, A\chi_S \rangle$$

Suppose we write $\chi_S = \sum_i b_i \overline{e_i}$. Then

$$b_1 = \langle \chi_S, \overline{e_i} \rangle = \langle \chi_S, \overline{u} + \overline{w} \rangle$$

$$= \frac{|S|}{\sqrt{n}} + \langle \chi_S, \overline{w} \rangle$$

$$= \frac{|S|}{\sqrt{n}} + O(\|\chi_S\| \|\overline{w}\|)$$

$$= \frac{|S|}{\sqrt{n}} + o(\sqrt{|S|})$$ by Step 2. Therefore,

$$|\langle \chi_S, A\chi_S \rangle - \lambda_1 b_1^2| \le |\lambda_2| \sum_i b_i^2 = |\lambda_2| \|\chi_S\|^2$$ and $$\left|2e(S) - pn\frac{|S|^2}{n} + o(pn|S|)\right| \le |\lambda_2||S|$$

i.e.,

$$\left|2e(S) - p|S|^2\right| = o(pn^2)$$

which is just DISC. ∎

Theorem:

For any family $\mathfrak{G}_p$ with $pn \to \infty$ as $n \to \infty$, the following implications hold for all $t \geq 1$:

$$\begin{array}{ccccccccc} \text{CIRCUIT}(2t) & \Leftrightarrow & \text{EIG}(2t) & \Rightarrow & \text{EIG} & \Rightarrow & \text{DISC} & \Leftrightarrow & \text{DISC}(1) \\ & & & & \Downarrow & & \Downarrow & & \\ & & & & \text{DISC}(t) & & \text{AR} & & \end{array}$$

What about the implications in the other directions ?

Mostly, they don't hold without further assumptions.

For example, AR $\not\Rightarrow$ DISC, and AR $\not\Rightarrow$ EIG are immediate.

Also, it is not hard to show that DISC(1) $\not\Rightarrow$ DISC(2).

However, it *is* true that DISC(2) $\Rightarrow$ DISC(1).

It is rather tricky to show that EIG $\not\Rightarrow$ EIG(4).

It can be done by first constructing a graph H with (about) $n^{1/5}$ vertices and $n^{3/10}$ edges, which has dominant eigenvalue of size about $n^{1/10}$, and all other eigenvalues of absolute value at most $cn^{1/20}$.
We then "blow up" H by replacing each vertex of H by $n^{4/5}$ new vertices which are connected "in the same way" that the original vertices in H were connected. This forms the graph $G = G_p(n)$ with $p \sim n^{-1/10}$.

It is now not too hard to show that G satisfies EIG but not EIG(4).

(This example was suggested by Tim Gowers).

In addition, it is clear that CIRCUIT(2t) $\not\Rightarrow$ CYCLE(2t) for every p, since it is well known that there exist graphs G(n) having $cn^{1+1/(2t-1)}$ edges but with girth greater than 2t.

It is clear that if we are to have a larger equivalence of quasi-random properties for sparse graphs, we will have to add some more restrictions.

Consider the following degree restriction property a graph family $\mathfrak{G}_p$ might have:

DEG :

For some absolute constant c, all vertices v of any $G(n) \in \mathfrak{G}_p$ satisfy

$$\boxed{\deg(v) < cpn.}$$

Theorem:

If $G(n)$ satisfies DEG then for any positive integer t,

$$\mathrm{DISC}(1) \Longrightarrow \mathrm{DISC}(t).$$

An even more stringent condition on a graph family $\mathfrak{G}_p$ is the following upper bound property on the walks of length t, where $t \geq 1$:

U(t): $\mathfrak{G}_p$ satisfies DEG, and for some constants c and c′, $p > cn^{-1+1/t}$

and for any $G(n) \in \mathfrak{G}_p$, and all pairs of vertices u,v in G(n),

$$e_t(u,v) < c' p^t n^{t-1}.$$

Note that any dense graph family (with p = c) satisfies U(1), and that U(s) $\Longrightarrow$ U(s+1) for any s.

Theorem: If $\mathfrak{G}_p$ satisfies U(t-1) for some $t \geq 2$, then

$$\mathrm{DISC}(t) \Longrightarrow \mathrm{CIRCUIT}(2t) \Longleftrightarrow \mathrm{CYCLE}(2t)$$

(The proofs are rather involved).

Putting this all together, we get the following cycle of implications:

$$
\begin{tikzcd}
 & \mathrm{AR} & \\
\mathrm{EIG}(2t) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \mathrm{EIG} \arrow[u, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathrm{DISC} \\
\mathrm{CIRCUIT}(2t) \arrow[d, Leftrightarrow, "U(t-1)"'] \arrow[dr, Leftrightarrow] & \mathrm{DISC}(1) \arrow[l, Rightarrow, "U(t-1)"'] \arrow[ur, Leftrightarrow] \arrow[d, Rightarrow, "U(t-1)"', "\mathrm{DEG}"] & \\
\mathrm{CYCLE}(2t) & \mathrm{DISC}(t) &
\end{tikzcd}
$$

<u>Theorem</u> : For any family $\mathfrak{G}_p$ satisfying U(t-1), $t \geq 2$,

the following conditions are equivalent:

DISC, DISC(1), DISC(t), CIRCUIT(2t), CYCLE(2t), EIG and EIG(2t).

We will refer to such graph families and graph properties as t-quasi-random.

Examples of t-quasi-random graph families

1. Erdös-Rényi graphs

q – prime power, $V(G_{ER})$ = points of a projective plane/GF(q)

Thus, $n = |V(G_{ER})| = q^2+q+1$.

Join (x, x', x'') to (y, y', y'') if $xy + x'y' + x''y'' = 0$ in GF(q).

In this case, edge density is $p = \frac{q+1}{q^2+q+1} \sim \frac{1}{2}n^{-\frac{1}{2}}$.

G_{ER} is not 2-quasi-random (e.g., it contains no 4-cycles)

but G_{ER} is 3-quasi-random.

In fact, much more is true of G_{ER}.

Let $X \subset V(G_{ER})$. Then

DISC $\Longrightarrow$ $\left| e(X) - p\binom{|X|}{2} \right| = o(pn^2) = o(n^{\frac{3}{2}})$.

The "truth" is $\left| e(X) - p\binom{|X|}{2} \right| = o(pn)^{\frac{1}{2}}|X| = o(n^{\frac{5}{4}})$.

As Thomason has pointed out, this stronger bound on edge density allows stronger conclusions to be drawn ("(p,α) – jumbled" graphs)

To get below $cn^{\frac{3}{2}}$ edges, more powerful constructions are needed.

Coset graphs (Chung)

p – prime, $t \geq 2$, an integer, g – a generator of $GF(p^t)^*$

$\lambda_g : GF(p^t)^* \to \{1, 2, \ldots, p^t - 1\}$ - the logarithm of $z \in GF(p^t)$

defined by $g^{\lambda_g(z)} = z$.

The graph $C_{p,t}$ has as its vertex set $\{1, 2, \ldots, p^t - 1\}$

For vertices u, v of $C_{p,t}$, $\{u, v\}$ is an edge of $C_{p,t}$ provided

$$u + v \in \{\lambda_g(y) : y \in \text{coset } g + GF(p)\}$$

<u>Theorem</u> $\{C_{p,t} : p \to \infty\}$ is t-quasi-random family.

(uses character sum estimates of Katz showing that $|\lambda_2| \leq 2(t-1)\sqrt{p}$).

Note: $C_{p,t}$ has $\sim \frac{1}{2} n^{1+\frac{1}{t}}$ edges and $n \sim p^t$ vertices.

Ramanujan graphs (Lubotsky, Phillips, Sarnak)

p, q – primes congruent to 1 mod 4, and with Legendre symbol $\left(\frac{p}{q}\right)=1.$

Then there exists a Ramanujan graph $X^{p,q}$ with the following parameters:

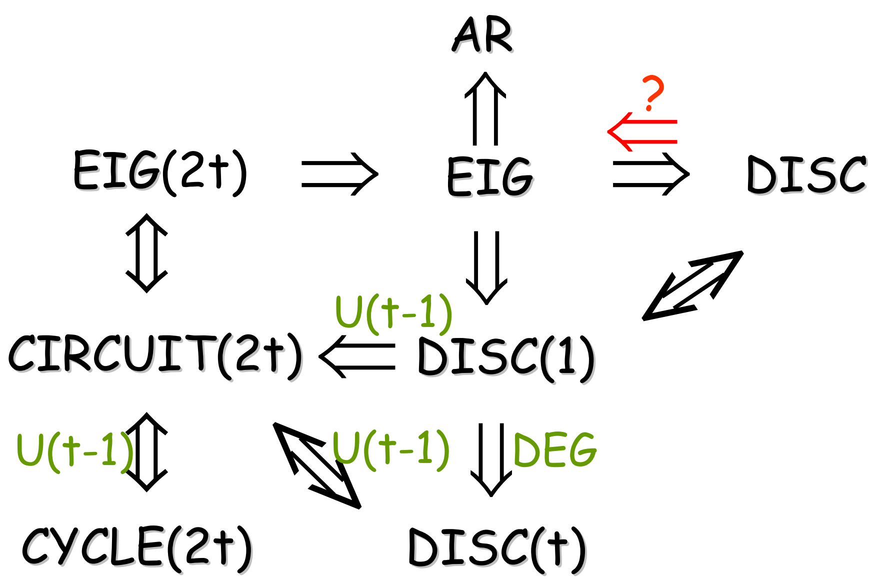
$$|V(X^{p,q})|=\tfrac{1}{2}q(q^2-1), \qquad |E(X^{p,q})|=\tfrac{1}{4}(p+1)q(q^2-1)$$

$$\lambda_1(X^{p,q})=p+1, \qquad |\lambda_2(X^{p,q})|\le 2\sqrt{p}\ .$$

This implies that $\{\, X^{p,q}:\ p,q\to\infty\}$ is t-quasi-random provided that $q^3\log q=o(p^t)$.

Some questions

1. Is it true that DISC $\Longrightarrow$ EIG ?



2. Is there a t-quasi-condition which depends on $\#\{H \prec G(n)\}$,

the number of induced copies of a fixed graph H in G(n) ?

Recall that for ordinary quasi-randomness we had

the induced s-subgraph property for $s \geq 4$, namely,

for every fixed graph H(s) on s vertices,

$$\#\{H(s) \prec G(n)\} = (1+o(1))n^s 2^{-\binom{s}{2}}$$

Thomason has shown that in this case if you have an error term for the edge discrepancy of significantly lower order than the main term, and $e(G(n)) > n^{2-\varepsilon_s}$ for sufficiently small $\varepsilon_s > 0$, then this holds.

3. Subgraph inclusion

Recall that for ordinary quasi-randomness the condition

$$\#\{C_4 \subset G(n)\} \leq (1+o(1))\frac{n^4}{16} \quad \text{and} \quad e(G(n)) \geq (1+o(1))\frac{n^2}{4}$$

guarantees quasi-randomness.

Conjecture ($100)

For any fixed bipartite graph H(s) which has a least one (non-trivial) cycle, the following property is quasi-random:

$$\#\{H \subset G(n)\} \leq (1+o(1))n^s 2^{-e(H)} \quad \text{and} \quad e(G(n)) \geq (1+o(1))\frac{n^2}{4}.$$

Perhaps the analogous conjecture holds for t-quasi-randomness if we assume that H(s) has girth at least 2t !

4. Quantitative versions of t-quasi-randomness

For a graph $G = G(n)$, define the deviation of G by:

$$\text{dev}(G) = \frac{1}{n^4}\,\text{Tr}(J - 2A(G))^4$$

where Tr denotes trace, J is the all 1's matrix, and $A(G)$ is the adjacency matrix of G. In particular, $0 \le \text{dev}(G) \le 1$.

For ordinary quasi-randomness with $p = \frac{1}{2}$, one can show that the closer dev(G) is to 0, the more each of the quasi-parameters of G is closer to their expected values. For example :

$$|\,e(G) - \frac{n^2}{4}\,| \le \frac{n^2}{4}\text{dev}(G)^{\frac{1}{4}},\ \ |\,\lambda_1(G) - \frac{n}{2}\,| \le \frac{n}{2}\text{dev}(G)^{\frac{1}{4}},\ \ |\,\lambda_2(G)\,| \le n\,\text{dev}(G)^{\frac{1}{16}}$$

and

$$|\#\{H(s) \prec G(n)\} - n^s 2^{-\binom{s}{2}}| \le 5n^s\,\text{dev}(G(n))^{\frac{1}{4}}$$

Is there an corresponding version of deviation for sparse graphs ??

5. What about Hypergraphs?

Many things are known, but so much more remains to be done !

(Chung, RLG, Rödl, Frankl, Gowers, ...)